EFFECTIVENESS FOR THE DUAL RAMSEY THEOREM

DAMIR D. DZHAFAROV, STEPHEN FLOOD, REED SOLOMON, AND LINDA WESTRICK

Abstract. We analyze the Dual Ramsey Theorem for k partitions and ℓ colors (DRT^k_ℓ) in the context of reverse math, effective analysis, and strong reductions. Over RCA_0, the Dual Ramsey Theorem stated for Baire colorings $\mathsf{Baire\text{-}DRT}^k_\ell$ is equivalent to the statement for clopen colorings ODRT^k_ℓ and to a purely combinatorial theorem CDRT^k_ℓ.

When the theorem is stated for Borel colorings and $k \geq 3$, the resulting principles are essentially relativizations of CDRT^k_ℓ. For each α, there is a computable Borel code for a Δ^0_α coloring such that any partition homogeneous for it computes $\emptyset^{(\alpha)}$ or $\emptyset^{(\alpha-1)}$ depending on whether α is infinite or finite.

For $k = 2$, we present partial results giving bounds on the effective content of the principle. A weaker version for Δ^0_n reduced colorings is equivalent to D^n_2 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$ and in the sense of strong Weihrauch reductions.

1. Introduction

This paper concerns the reverse mathematical and computational strength of variations of the Dual Ramsey Theorem. For $k \leq \omega$, let $(\omega)^k$ denote the set of all partitions of ω into exactly k pieces. Such a partition can be represented as a surjective function from ω to k. Thus $(\omega)^k$ inherits a natural topology by considering it as a subset of k^ω.

Dual Ramsey Theorem ([4], [14])**.** *For any $k, \ell < \omega$, suppose we have a coloring $(\omega)^k = \cup_{i<\ell} C_i$. If for each $i < \ell$, C_i has the property of Baire, then there is a partition $p \in (\omega)^\omega$ such that any coarsening of p down to exactly k pieces has the same color.*

The reason that this theorem is dual to the original Ramsey's Theorem concerns what objects are being colored. In the original Ramsey's theorem, we color the k-element subsets of ω, which correspond to injective functions from k to ω. In the Dual Ramsey Theorem, we color surjective functions from ω to k.

A straightforward choice argument shows that the Dual Ramsey Theorem fails if no regularity conditions on the C_i are assumed. The theorem was first proved for Borel colorings by Carlson and Simpson [4], and extended to colorings with the Baire property by Prömel and Voigt [14]. From the perspective of reverse mathematics or computational mathematics, the variation in hypothesis gives us two theorems to consider. We call them the *Borel Dual Ramsey Theorem* and the *Baire Dual Ramsey Theorem* respectively.

Carlson and Simpson asked for a recursion-theoretic analysis of the Borel Dual Ramsey Theorem. In order to answer this, it is necessary to choose a method for

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encoding the coloring, and one must consider the potential effects of a topologically intricate coloring. Previous work side-stepped these issues by restricting attention to open colorings only [12] or by focusing attention only on the main combinatorial lemma which Carlson and Simpson used in their proof, and on its variable word variants [12, 7, 11].

From the work of [12], we know that over RCA_0, ODRT^k_ℓ implies RT^{k-1}_ℓ, where ODRT^k_ℓ is the restriction of the Borel Dual Ramsey Theorem to open colorings only, and RT is the usual Ramsey's Theorem. This provides a lower bound on the strength of the Borel Dual Ramsey Theorem. Conversely, in unpublished work Slaman has shown that the Borel Dual Ramsey Theorem follows from Π^1_1-CA_0 [18]. No direct implication is known between the Dual Ramsey Theorems and the variable word theorems, because the Dual Ramsey Theorem does not require the "words" in its solution to be finite (and by Proposition 3.15, it cannot require this), while the proof of the Dual Ramsey Theorem from the variable word theorems uses infinitely many sequential applications of the latter (Theorem 3.18). Overall, this leaves a rather large gap, and we do not close it. However, we do provide significant clarification of the key difficulties. In particular, for the first time we directly tackle the topological aspect of the Borel version of the theorem.

1.1. **Combinatorial core of the Borel Dual Ramsey Theorem.** Since the Borel version follows from the Baire version plus the additional principle "Every Borel set has the property of Baire", our first step is to understand the Baire version.

To be clear, an instance of the Baire Dual Ramsey Theorem is a sequence of pairwise disjoint open sets $O_0, \ldots, O_{\ell-1}$ whose union is dense in $(\omega)^k$, and a sequence of dense open sets $\{D_n\}_{n\in\omega}$. Such an instance simultaneously represents all colorings $(\omega)^k = \cup_{i<\ell} C_i$ for which the symmetric difference $C_i \Delta O_i$ is disjoint from $\cap_n D_n$. There may be uncountably many such colorings, because no condition is placed on how $2^\omega \setminus \cap_n D_n$ is colored. Any solution $p \in (\omega)^\omega$ to the Baire version must have $(p)^k \subseteq \cap_n D_n$.

In Section 3.1 we define a purely combinatorial principle CDRT^k_ℓ, which precisely captures the strength of the Baire version. In the following, if $p \in (\omega)^\omega$ and $k \le \omega$, let $(p)^k$ denote the set of coarsenings of p into exactly k pieces. Recalling that we consider p as a surjective function $p : \omega \to k$, let

$$p^* := p \upharpoonright \min p^{-1}(k-1).$$

In other words, p^* is a string on alphabet $k-1$, it tells us by its length what is the smallest element of p's last block, and it tells us how p partitions the finitely many smaller elements into its first $k-1$ blocks. Let $(<\omega)^{k-1} = \{p^* : p \in (\omega)^k\}$.

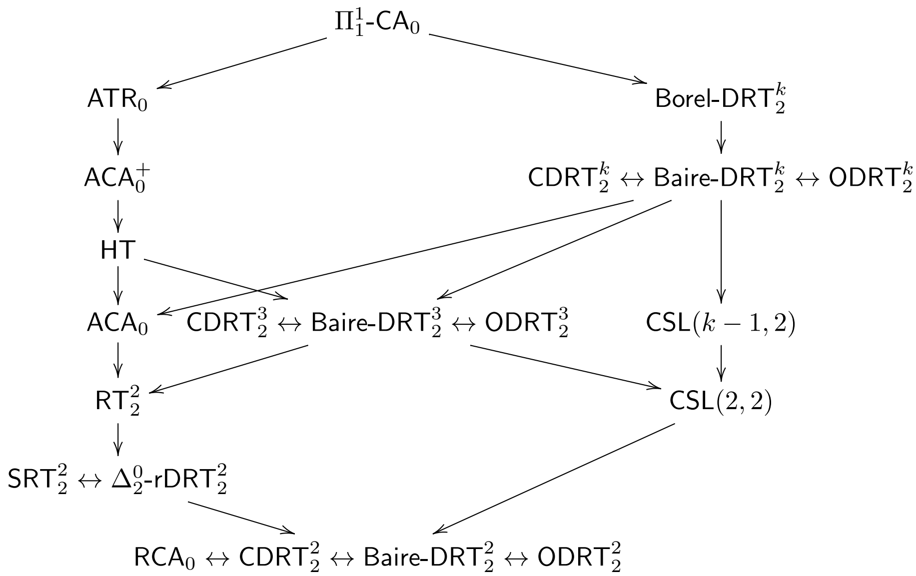
Theorem 1.1. *Let $k, \ell < \omega$. Over RCA_0, the following are equivalent.*

1. *The Baire Dual Ramsey Theorem for k partitions and ℓ colors.*
2. ODRT^k_ℓ
3. CDRT^k_ℓ*, which states: for every $c : (<\omega)^{k-1} \to \ell$, there is a $p \in (\omega)^\omega$ and a color $i < \ell$ such that for every $x \in (p)^k$, $c(x^*) = i$.*

Thus we have reduced the Baire version of the theorem to a purely combinatorial statement. The proof of the equivalence is essentially an effectivization of [14].

Aside from the results in [12], the strengths of the CDRT^k_ℓ statements are wide open. We include one more result, which was known to Simpson (see [4, page 268])

FIGURE 1. Implications over RCA_0 between variants of the Dual Ramsey Theorem considered in this paper and some related principles. The parameter $k \geq 4$ is arbitrary.



and subsequently rediscovered by Patey [13]: a proof of one case of the Carlson-Simpson Lemma from Hindman's Theorem. With minor modifications, we adapt this proof in Section 3.2 to show that Hindman's Theorem for ℓ colorings implies the stronger CDRT^3_ℓ. See Figure 1.1 for a summary of what is known about the combinatorial core of the Dual Ramsey Theorem.

We close Section 3 with a self-contained proof of CDRT^k_ℓ from the Carlson-Simpson Lemma (Theorem 3.18). In our proof, the only non-constructive steps are $\omega \cdot (k-2)$ nested applications of the Carlson-Simpson Lemma.

The earliest claim we are aware of for a proof of CDRT^k_ℓ is in [14], where a generalization of CDRT^k_ℓ called *Theorem A* is attributed to a preprint of Voigt titled "Parameter words, trees and vector spaces". However, as far as we can tell, this paper never appeared. Another proof of CDRT^k_ℓ can be found in [19], but as a corollary of a larger theory.

1.2. **Computational strength of the Borel Dual Ramsey Theorem.** In Sections 4 and 5, we consider the Borel Dual Ramsey Theorem, or Borel-DRT, from the perspective of effective combinatorics. The behavior is different depending on the number of pieces k in the partition, with the $k \geq 3$ case being addressed in Section 4 and the $k = 2$ case in Section 5.

When $k \geq 3$, given a fast-growing function f one can design an open, f-computable coloring such that all of its homogeneous partitions compute a function which dominates f (this was already essentially done in [12]). But if f is hyperarithmetic, that same coloring has an effective Borel code as a Δ^0_α set. Thus by sneaking the computation of f into an effective Borel code, we obtain a computable instance of $\mathsf{Borel\text{-}DRT}^3_2$. As a result, $\mathsf{Borel\text{-}DRT}^3_2$ can be informally considered as some kind of hyperjump of ODRT^3_2. Formally, we have the following in Theorem 4.7.

Theorem 1.2. *For every computable ordinal $\alpha > 0$ and every $k \geq 3$, there is a computable Borel code for a Δ^0_α coloring $c : (\omega)^k \to 2$ such that every infinite partition homogeneous for c computes $\emptyset^{(\alpha)}$ if α is infinite, or $\emptyset^{(\alpha-1)}$ if α is finite.*

The preceding theorem gives a coding lower bound on the complexity of solutions for $k \geq 3$. In contrast, we remark that the best known basis theorem for the $k \geq 3$ case is still the following result of Slaman [18]: Every hyperarithmetic instance of the Borel Dual Ramsey Theorem has a hyperarithmetically low solution. This result can also be extracted from our analysis as follows. Given a Borel coloring, there is a hyperarithmetic witness that it has the property of Baire. Use Theorem 1.1 to computably reduce this instance of the Baire Dual Ramsey Theorem to an instance of CDRT. It is arithmetic to check whether a given partition $p \in (\omega)^\omega$ is a solution to a given instance c of CDRT. Therefore the collection of solutions is non-empty Σ^1_1. Applying the Gandy Basis Theorem gives the desired solution.

When $k = 2$, it is likewise possible to create effectively Borel instances which correspond to hyperarithmetically computable open colorings. However, there are two important differences with the $k = 2$ case. First, ODRT^2_ℓ is computably true. As a consequence, when $k = 2$ the Borel variant has a sharper basis theorem.

Theorem 1.3. *Every Δ^0_n instance of $\mathsf{Borel\text{-}DRT}^2_\ell$ has a Δ^0_n solution.*

This result follows from the more general Theorem 5.4. Note that the Δ^0_n instance is a subset of $(\omega)^k$ which could be topologically intricate, while the solution is a single Δ^0_n partition $p \in (\omega)^\omega$.

The second difference in the $k = 2$ case is that CDRT^2_ℓ is Weihrauch equivalent to the infinite pigeonhole principle RT^1_ℓ. (Observe that an instance of CDRT^2_ℓ is essentially a coloring of ω.) This immediately offers lower bounds: for each n, $D^n_\ell \leq_{\mathrm{sW}} \mathsf{Borel\text{-}DRT}^2_\ell$, where D^n_ℓ is the problem whose instances are Δ^0_n colorings $c : \omega \to \ell$ and whose solutions are the infinite sets monochromatic for c. The question is whether these could possibly be equivalences when $\mathsf{Borel\text{-}DRT}^2_\ell$ is likewise restricted to Δ^0_n instances. We are only able to show a partial result in this direction (Theorem 5.7).

Theorem 1.4. *Let Δ^0_n-rDRT^2_2 be the restriction of $\mathsf{Borel\text{-}DRT}^2_2$ to instances c which are given by Δ^0_n formulas and for which c is* reduced, *meaning that $c(p)$ depends only on p^* for all $p \in (\omega)^2$. Then*

1. Δ^0_n-$\mathsf{rDRT}^2_2 \equiv_{\mathrm{sW}} \mathsf{D}^n_2$.
2. *Over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$, Δ^0_n-rDRT^2_2 is equivalent to D^n_2.*

1.3. **Reverse mathematics and Borel sets.** In Section 6, we consider problems motivated by the reverse mathematics of the Borel Dual Ramsey Theorem. We observe that the Borel Dual Ramsey Theorem can be obtained by composing "Every Borel set has the property of Baire" (let us call it BP) with the Baire Dual Ramsey Theorem. So a natural next step is to understand the strength of BP. We show the following as a part of Theorem 6.9.

Theorem 1.5. *Over RCA_0, ATR_0 is equivalent to the following statement. For every Borel code B, there is some point x such that $x \in B$ or $x \notin B$.*

This result mainly shows that the usual definition of Borel sets, which is given in [17] using ATR_0 as a base theory, really does not make sense in the absence of ATR_0. This provides an obstacle to a satisfactory analysis of BP. While BP follows from ATR_0, (Proposition 6.5), in the reversal BP formally implies ATR_0 only due to the technical reason above. We leave a deeper analysis of BP and the Borel Dual Ramsey Theorem to future work [2].

The proof of Theorem 1.5 uses a method of effective transfinite recursion, ETR, which is available in ACA_0 (and possibly in weaker systems). Greenberg and Montalbán [8] use ETR to establish equivalences of ATR_0 and claim that ETR is provable in RCA_0. However, their proof of ETR overlooks an application of Σ^0_1 transfinite induction, and in general, transfinite induction for Σ^0_1 formulas does not hold in RCA_0. While the main results in [8] continue to hold because Greenberg and Montalbán show the classified theorems imply ACA_0 without reference to ETR (and hence can use ETR in ACA_0 to complete the equivalence with ATR_0), we have included a proof of ETR in Section 6 to make explicit the use of transfinite induction.

In the final Section 7 we list a number of open questions.

2. Notation

We use ω to denote the natural numbers, which in subsystems of Z_2 is the set $\{x : x = x\}$, often denoted by $\mathbb{N}$ in the literature. Despite this notation, we do not restrict ourselves to ω-models. Second, when we refer to the parameters k and ℓ in versions of the Dual Ramsey Theorem, we assume k and ℓ are arbitrary standard numbers with $k, \ell \geq 2$. By a statement such as "RCA_0 proves $\mathsf{Borel\text{-}DRT}^k_\ell$ implies $\mathsf{Baire\text{-}DRT}^k_\ell$", we mean, for all $k, \ell \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{Borel\text{-}DRT}^k_\ell \rightarrow \mathsf{Baire\text{-}DRT}^k_\ell$. For many results, the quantification over k and ℓ can be pulled inside the formal system. However, in some cases, issues of induction arise and we wish to set those aside in this work.

For $k \leq \omega$, let $k^{<\omega}$ denote the set of finite strings over k and let k^ω denote the set of functions $f : \omega \rightarrow k$. As noted above, unless explicitly stated otherwise, we will always assume that $k \geq 2$. For $\sigma \in k^{<\omega}$, $|\sigma|$ denotes the length of σ, and if $|\sigma| > 0$, $\sigma(0), \ldots, \sigma(|\sigma| - 1)$ denote the entries of σ in order. For $p \in k^\omega$ and $\sigma \in k^{<\omega}$, we write $\sigma \prec p$ if σ is an initial segment of p. Similarly, if $\sigma, \tau \in k^{<\omega}$, we write $\sigma \preceq \tau$ if σ is an initial segment of τ and $\sigma \prec \tau$ if σ is a proper initial segment of τ. We write $p \upharpoonright n$ to denote the string obtained by restricting the domain of p to n. The standard (product) topology on k^ω is generated by basic clopen sets of the form

$$[\sigma] = \{p \in k^\omega : \sigma \prec p\}$$

for $\sigma \in k^{<\omega}$.

We use the following notational conventions for partitions. For $k \leq s \leq \omega$, we use $(s)^k$ to denote the set of all partitions of s into exactly k pieces. The pieces are also called *blocks*. Each such partition can be viewed as a surjective function $p : s \rightarrow k$, where the blocks are the sets $p^{-1}(i)$ for $i < k$. More than one surjective function can describe the same partition, so we pick a canonical one. We say that $p : s \rightarrow k$ is *ordered* if for each $i < j < k$, $\min p^{-1}(i) < \min p^{-1}(j)$. We then more formally define the *k-partitions of s* as

$$(s)^k = \{p \in k^s : \ p \text{ is surjective and ordered}\}.$$

We also let $(<\omega)^k$ denote $\cup_{r \in \omega} (r)^k$.

If $k \leq s \leq t \leq \omega$ and $p \in (t)^s$, then we define $(p)^k = \{x \circ p : x \in (s)^k\}$. In English, if p is a partition of t into exactly s pieces, $(p)^k$ is the set of ways to further coarsen t down to exactly k pieces, so we call $(p)^k$ the set of *k-coarsenings* of p.

If $(\omega)^k = \cup_{i<\ell} C_i$ and $p \in (\omega)^\omega$ with $(p)^k \subseteq C_i$, then we say that p is *homogeneous* for the color C_i.

The set $(\omega)^k$ inherits the subspace topology from k^ω with basic open sets of the form $[\sigma] \cap (\omega)^k$ for $\sigma \in k^{<\omega}$. This topology is also natural from the partition perspective. For example, if we considered a partition instead as an equivalence relation $R \subseteq \omega \times \omega$, the same topology is also generated by declaring $\{R : (n,m) \in R\}$ to be clopen for each pair $(n,m) \in \omega \times \omega$.

The space $(\omega)^k$ is not compact since, for example, the collection of open sets $[0^n 1]$ for $n \geq 1$ cover $(\omega)^2$ but this collection has no finite subcover. However, if $\sigma \in (<\omega)^k$, then $[\sigma] \subseteq (\omega)^k$ and $[\sigma]$ is a compact clopen subset of $(\omega)^k$. To generate the topology on $(\omega)^k$, it suffices to restrict to the basic clopen sets of the form $[\sigma]$ with $\sigma \in (<\omega)^k$. Although the notation $[\sigma]$ is ambiguous about whether the ambient space is k^ω or $(\omega)^k$ (or ℓ^ω or $(\omega)^\ell$ for some $\ell > k$), the meaning will be clear from context.

We denote the ith block of the partition p by $p^{-1}(i)$ (we start counting the blocks at 0, so the last block of a k-partition is indexed by $i = k-1$). We denote the least element of $p^{-1}(i)$ by $\mu^p(i)$. If $p \in (\omega)^k$, we will often have use for the string $p^* = p \upharpoonright \mu^p(k-1)$. We can also apply this notation if $p \in (s)^k$ for any $s \geq k$.

Sometimes it is convenient to consider colorings of $(p)^k$ for some $p \in (\omega)^\omega$, and then ask for a homogeneous partition $q \in (p)^\omega$. This is not really more general than the case we have been considering, because a coloring $(p)^k = \cup_{i<\ell} C_i$ corresponds canonically to the coloring of $(\omega)^k$ defined by

$$x \in \widehat{C}_i \iff x \circ p \in C_i. \tag{1}$$

In this case any $y \in (\omega)^\omega$ is homogeneous for $\{\widehat{C}_i\}_{i<\ell}$ if and only if $y \circ p$ is homogeneous for $\{C_i\}_{i<\ell}$.

3. The Baire Dual Ramsey Theorem

3.1. **Three versions of the Baire Dual Ramsey Theorem.** We formulate three versions of the Baire Dual Ramsey Theorem in second order arithmetic and show they are equivalent over RCA_0.

Coding colorings or sets with the Baire property in second order arithmetic is complicated by the fact that there are $2^{\mathfrak{c}}$ (where $\mathfrak{c} = 2^{\aleph_0}$) many subsets of $(\omega)^k$ or k^ω with the Baire property. However, if we identify colorings which are the same after discarding a meager set, then there are only continuum many with the Baire property. Specifying only an equivalence class of colorings is consistent with how theorems which hypothesize the Baire property usually work. They start by fixing a comeager approximation to the set in question and then proceed by working exclusively with this approximation. This classical observation motivates our definition of a code for a Baire coloring.

Definition 3.1 (RCA_0)**.** A *code for an open set in* $(\omega)^k$ is a set $O \subseteq \omega \times (<\omega)^k$. We say that a partition $p \in (\omega)^k$ is *in the open set coded by* O (or just *in* O and write $p \in O$) if there is a pair $\langle n, \sigma\rangle \in O$ such that $p \in [\sigma]$.

A *code for an closed set in* $(\omega)^k$ is also a set $V \subseteq \omega \times (<\omega)^k$. In this case, we say $p \in (\omega)^k$ *is in* V (and write $p \in V$) if for all pairs $\langle n, \sigma\rangle \in V$, $p \notin [\sigma]$.

Definition 3.2 (RCA_0)**.** An open set $O \subseteq (\omega)^k$ is *dense* if for all $\tau \in (<\omega)^k$, $[\tau] \cap O \neq \emptyset$. That is, for all τ, there is a pair $\langle n, \sigma\rangle \in O$ such that σ and τ are comparable as strings.

Definition 3.3 (RCA_0)**.** A *code for a Baire ℓ-coloring of* $(\omega)^k$ is a sequence of dense open sets $\{D_n\}_{n<\omega}$ together with a sequence of pairwise disjoint open sets $\{O_i\}_{i<\ell}$ such that $\bigcup_{i<\ell} O_i$ is dense in $(\omega)^k$.

Recall that RCA_0 suffices to prove the Baire Category Theorem: if $\{D_n\}_{n<\omega}$ is a sequence of dense open sets, then $\cap_{n<\omega} D_n$ is dense. Classically, if a coloring $\cup_{i<\ell} C_i = (\omega)^k$ has the Baire property, then it has a comeager approximation given by sequences of open sets $\{O_i\}_{i<\ell}$ and $\{D_n\}_{n<\omega}$ such that each D_n is dense and for each $p \in \cap_{n<\omega} D_n$, $p \in C_i$ if and only if $p \in O_i$.

We abuse terminology and refer to the Baire code as a *Baire ℓ-coloring* of $(\omega)^k$. Similarly, an *open ℓ-coloring* is a coloring $(\omega)^k = \cup_{i<\ell} O_i$ in which the O_i are open and pairwise disjoint.

Definition 3.4. For each (standard) $k, \ell \geq 2$, we define $\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ and CDRT^k_ℓ in RCA_0 as follows.

1. $\mathsf{Baire\text{-}DRT}^k_\ell$: For every Baire ℓ-coloring $\{O_i\}_{i<\ell}$ and $\{D_n\}_{n<\omega}$ of $(\omega)^k$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $x \in O_i \cap \bigcap_n D_n$.
2. ODRT^k_ℓ: For every open ℓ-coloring $(\omega)^k = \cup_{i<\ell} O_i$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $x \in O_i$.
3. CDRT^k_ℓ: For every coloring $c : (<\omega)^{k-1} \to \ell$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $c(x^*) = i$.

Our first goal is to show that the instances of CDRT^k_ℓ are in one-to-one canonical correspondence with those instances of ODRT^k_l for which the coloring of $(\omega)^k$ is *reduced*. We define a reduced coloring without considering the coding method and note that any reduced coloring is open.

Definition 3.5. Let $y \in (\omega)^\omega$ and $m < k$. A coloring of $(y)^k$ is *m-reduced* if whenever $p, q \in (y)^k$ and $p \upharpoonright \mu^p(m) = q \upharpoonright \mu^q(m)$, p and q have the same color. A coloring of $(y)^k$ is *reduced* if it is $(k-1)$-reduced.

Note that a coloring is reduced means that the color of each partition $p \in (y)^k$ depends only on p^*.

Proposition 3.6 (RCA_0)**.** *The following are equivalent.*

1. CDRT^k_ℓ.
2. *For every open reduced coloring $(\omega)^k = \cup_{i<\ell} O_i$, there are $p \in (\omega)^\omega$ and $i < \ell$ such that $(p)^k \subseteq O_i$.*
3. *For every $y \in (\omega)^\omega$ and open reduced coloring $(y)^k = \cup_{i<\ell} O_i$, there are $p \in (y)^\omega$ and $i < \ell$ such that $(p)^k \subseteq O_i$.*

Proof. Clearly (3) implies (2). To see that (2) implies (3), fix $y \in (\omega)^\omega$ and a reduced open coloring $(y)^k = \cup_{i<\ell} O_i$. Define

$$\widehat{O}_i = \{\langle n, \tau\rangle : \tau \in (<\omega)^k \text{ and } \tau \circ y \in O_i\}$$

It is straightforward to check that the coloring $(\omega)^k = \cup_{i<\ell} \widehat{O}_i$ is also reduced, and that whenever x is homogeneous for $\cup_{i<\ell} \widehat{O}_i$ then $x \circ y$ is homogeneous for $\cup_{i<\ell} O_i$.

To see (2) implies (1), fix $c : (<\omega)^{k-1} \to \ell$. For each $i < \ell$, let

$$O_i = \{\langle 0, \sigma^\frown(k-1)\rangle : \sigma \in (<\omega)^{k-1} \text{ and } c(\sigma) = i\}.$$

Then $(\omega)^k = \cup_{i<\ell} O_i$ is an open reduced coloring of $(\omega)^k$, and any infinite partition which is homogeneous for it is also homogeneous for c.

For the implication from (1) to (2), assume CDRT^k_ℓ, and suppose we are given a coloring $\cup_{i<\ell} O_i$. Now, for each $\sigma \in (<\omega)^{k-1}$, we define $c(\sigma)$ as follows. Note that for some $i < \ell$, some $\tau \succeq \sigma^\frown(k-1)$, and some n, we have $\langle n, \tau\rangle \in O_i$. Letting $\langle n, \tau, i\rangle$ be the least triple with this property, we define $c(\sigma) = i$.

Let $i < \ell$ and $p \in (\omega)^\omega$ be the result of applying CDRT^k_ℓ to c. Given $x \in (p)^k$, we know that $c(x^*) = i$. Let n, τ be the witnesses used in the definition of $c(x^*)$. Let $q \in (\omega)^k$ with $q \succ \tau$. Then $q \in O_i$. Since O_i is reduced and $q^* = \tau^* = x^*$, $x \in O_i$. Therefore, p is homogeneous for the coloring $\cup_{i<\ell} O_j$, as required. □

It is now routine to show that the number of colors does not matter.

Proposition 3.7 (RCA_0)**.** *CDRT^k_ℓ and CDRT^k_2 are equivalent.*

Proof. Collapse colors and iterate CDRT^k_2 finitely many times, using Proposition 3.6. □

The next proof is essentially an effective version of an argument in [14].

Theorem 3.8 (RCA_0)**.** *$\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ and CDRT^k_ℓ are equivalent.*

Proof. By setting $D_n = (\omega)^k$ in $\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ is a special case of $\mathsf{Baire\text{-}DRT}^k_\ell$, and by Proposition 3.6, CDRT^k_ℓ is a special case of ODRT^k_ℓ. It remains to prove in RCA_0 that CDRT^k_ℓ implies $\mathsf{Baire\text{-}DRT}^k_\ell$.

Let $\{O_i\}_{i<\ell}$, $\{D_n\}_{n<\omega}$ be a Baire ℓ-coloring of $(\omega)^k$ for which the open sets O_i are pairwise disjoint. We construct a partition $y \in (\omega)^\omega$ such that $(y)^k \subseteq \cap_n D_n$ and $\cup_i O_i$ restricted to $(y)^k$ is reduced. By Proposition 3.6 and CDRT^k_ℓ, there is a homogeneous $z \in (y)^\omega$ for this open reduced coloring. Since $(z)^k \subseteq (y)^k \subseteq \cap_n D_n$, this partition z is homogeneous for the original Baire coloring.

First we describe the construction in a classical way, and then remark on how it can be carried out in RCA_0.

Build y by initial segments in stages, $y = \lim_s y_s$, starting with y_0 being the empty string, and then continuing with stage $s = 1$ as follows. Assume that at the start of stage s, y_{s-1} is an $(s-1)$-partition. In stage s begin by letting $y^0_s = y_{s-1}{}^\frown(s-1)$, so that y^0_s is an s-partition. Let $x_0, \ldots, x_r$ be a list of the elements of $(s)^k$. For each $i = 0, \ldots r$, let $q = x_i \circ y^i_s$. Let $\tau \in (<\omega)^k$ be such that $q \preceq \tau$ and τ meets $\cap_{n\leq s} D_n$ and $\cup_{i<\ell} O_i$. Then extend y^i_s to y^{i+1}_s in such a way that $x_i \circ y^{i+1}_s = \tau$. In general there is more than one way to do this, but which way does not matter. For concreteness, for each $n \geq |y^i_s|$ we could set $y^{i+1}_s(n)$ to be the least m such that $x_i(m) = \tau(n)$. At the conclusion of these substages we are left with y^{r+1}_s. Let $y_s = y^{r+1}_s$. This completes the construction of y.

We need to justify why this construction can be carried out in RCA_0. To that end, we make the following claims in RCA_0:

(1) For any $q \in (<\omega)^k$ and s, there is an extension $\tau \succeq q$ which meets $\cup_{i<\ell} O_i$ and $\cap_{n\leq s} D_n$. To see that for all s, such a τ exists, apply Σ^0_1 induction.
(2) There is a function $f : (<\omega)^k \times \omega \to (<\omega)^k$ with the properties above. This follows because in RCA_0, we can select the τ with least witness.

(3) There is a function which outputs the sequence

$$y_1^0, \dots, y_1^{r_1}, y_2^0, \dots, y_2^{r_2}, y_3^0, \dots$$

This can be obtained by primitive recursion using the function f.

Therefore, y exists in RCA_0. Next we show that $(y)^k \subseteq \cap_n D_n$. Let $w \in (y)^k$ and fix n. Let $x \in (\omega)^k$ with $x \circ y = w$. Let $s \geq n$ be large enough that $x \upharpoonright s \in (s)^k$. Then $x \upharpoonright s$ was one of the ways to coarsen considered during stage s of the construction. By construction, $(x \upharpoonright s) \circ y_s$ meets D_n. So $x \circ y \in D_n$.

Finally, we claim that the restriction of $\cup_{i<\ell} O_i$ to $(y)^k$ is a reduced coloring. Given if $w_1, w_2 \in (y)^k$ with $w_1^* = w_2^*$, let x_1 and x_2 be such that $x_1 \circ y = w_1$ and $x_2 \circ y = w_2$. Then $x_1^* = x_2^*$. Let $x = (x_1^*)^\frown(k-1)$ and let $s = |x|$. Then $x \in (s)^k$ and x was considered at stage s of the construction. By construction, $x \circ y_s$ meets O_i for some i. Since $x \circ y_s$ is an initial segment of both w_1 and w_2, it follows that w_1 and w_2 are both in O_i. Finally, as $w_1, w_2 \in \cap_n D_n$, we have $w_1, w_2 \in C_i$, as needed. □

Since ODRT^{k+1}_ℓ implies RT^k_ℓ over RCA_0 [12], we have the following corollary.

Corollary 3.9 (RCA_0)**.** CDRT^{k+1}_ℓ *implies* RT^k_ℓ.

Proposition 3.10. *For any $\ell \geq 2$, RCA_0 proves CDRT^2_ℓ and hence also ODRT^2_ℓ.*

Proof. Let $c : (<\omega)^1 \to \ell$. Since $(<\omega)^1 = \{0^n : n \in \omega\}$, c can be viewed as an ℓ-coloring of ω. By RT^1_ℓ, there is a color i and an infinite set X such that for every $n \in X$, $c(0^n) = i$. Let z be the partition which has a block of the form $\{n\}$ for each $n \in X$ and puts all the other numbers in $z^{-1}(0)$. Then z is homogeneous for c. □

3.2. **Connections to Hindman's theorem.** In this section, we show that Hindman's Theorem for ℓ-colorings implies CDRT^3_ℓ. In [4], Simpson remarks that one case of the Carlson-Simpson Lemma follows from Hindman's Theorem. Ludovic Patey showed us a proof, and the same argument gives a strong form of CDRT^3_ℓ. We include Patey's proof here.

Definition 3.11 (RCA_0)**.** Let $\mathcal{P}_{\text{fin}}(\omega)$ denote the set of (codes for) all non-empty finite subsets of ω. $X \subseteq \mathcal{P}_{\text{fin}}(\omega)$ is an *IP set* if X is closed under finite unions and contains an infinite sequence of pairwise disjoint sets.

Theorem 3.12 (Hindman's theorem for ℓ-colorings)**.** *For every $c : \mathcal{P}_{fin}(\omega) \to \ell$ there is an IP set X and a color $i < \ell$ such that $c(F) = i$ for all $F \in X$.*

Theorem 3.13 (essentially Patey [13], see also [4, page 268])**.** *Over RCA_0, Hindman's theorem for ℓ-colorings implies CDRT^3_ℓ. In particular, CDRT^3_ℓ is provable in ACA_0^+.*

Proof. Hindman's Theorem follows from ACA_0^+ by [3], so it suffices to prove the first statement. Fix $\ell \geq 2$ and assume Hindman's Theorem for ℓ-colorings. Since Hindman's Theorem for 2-colorings implies ACA_0, we reason in ACA_0. By Proposition 3.6, it suffices to fix an open reduced coloring $(\omega)^3 = \cup_{i<\ell} O_i$ and produce $p \in (\omega)^\omega$ and $i < \ell$ such that for all $x \in (p)^3$, $x \in O_i$. We write the coloring as $c : (\omega)^3 \to \ell$ with the understanding that $c(x) = i$ is shorthand for $x \in O_i$.

For a nonempty finite set $F \subseteq \omega$ with $0 \notin F$ and a number $n > \max F$, we let $x_{F,n} \in (\omega)^3$ be the following partition.

$$x_{F,n}(k) = \begin{cases} 0 & \text{if } k \notin F \text{ and } k \neq n \\ 1 & \text{if } k \in F \\ 2 & \text{if } k = n \end{cases}$$

Thus, the blocks are $\omega - (F \cup \{n\})$, F and $\{n\}$. Note that we can determine the color $c(x_{F,n})$ as a function of F and n and that since c is reduced, if $x \in (\omega)^3$ and $x \upharpoonright \mu^x(2) = x_{F,n} \upharpoonright n$, then $c(x) = c(x_{F,n})$.

The remainder of the proof is most naturally presented as a forcing construction. After giving a classical description of this construction, we indicate how to carry out the construction in ACA_0. The forcing conditions are pairs (F, I) such that

- F is a non-empty finite set such that $0 \notin F$,
- I is an infinite set such that $\max F < \min I$, and
- for every nonempty subset U of F there is an $i < \ell$ such that $c(x_{U,n}) = i$ for all $n \in F \cup I$ with $\max U < n$.

Extension of conditions is defined as for Mathias forcing: $(\widehat{F}, \widehat{I}) \leq (F, I)$ if $F \subseteq \widehat{F} \subseteq F \cup I$ and $\widehat{I} \subseteq I$.

By the pigeonhole principle, there is an $i < l$ such that $c(x_{\{1\},n}) = i$ for infinitely many $n > 1$. For any such i, the pair $(\{1\}, \{n \in \omega : n > 1 \text{ and } c(x_{\{1\},n}) = i\})$ is a condition. More generally, given a condition (F, I) there is an infinite set $\widehat{I} \subseteq I$ such that $(F \cup \{\min I\}, \widehat{I})$ is also a condition. To see this, let $U_0, \ldots, U_{s-1}$ be the nonempty subsets of $F \cup \{\min I\}$ containing $\min I$. By arithmetic induction, for each positive $k \leq s$, there exist colors $i_0, \ldots, i_{k-1} < \ell$ such that there are infinitely many $n \in I$ with $c(x_{U_j,n}) = i_j$ for all $j < k$. (If not, fix the least k for which the fact fails, and apply the pigeonhole principle to obtain a contradiction.) Let $i_0, \ldots, i_{s-1}$ be the colors corresponding to $k = s$ and let $\widehat{I}$ be the infinite set $\{n \in I : \forall j < s \; (c(x_{U_j,n}) = i_j)\}$.

Fix a sequence of conditions $(F_1, I_1) > (F_2, I_2) > \cdots$ with $|F_k| = k$ and let $G = \bigcup_k F_k$. To complete the proof, we use G to define a coloring $d : \mathcal{P}_{\text{fin}}(\omega) \to \ell$ to which we can apply Hindman's Theorem. However, first we indicate why we can form G in ACA_0.

The conditions (F, I) used to form G can be specified by the finite set F, the number $m = \min I$ and the finite sequence $\delta \in \ell^M$ where $M = 2^{|F|} - 1$ such that if $F_0, \ldots, F_{M-1}$ is a canonical listing of the nonempty subsets of F, then $I = \{n \geq m : \forall j < M \, (c(x_{F_j,n}) = \delta(j))\}$. The extension procedure above can be captured by an arithmetically definable function $f(F, m, \delta) = \langle F \cup \{m\}, m', \delta' \rangle$ where $F \cup \{m\}$, m' and δ' describe the extension $(F \cup \{m\}, \widehat{I})$. Because the properties of this extension where verified using arithmetic induction and the pigeonhole principle, both of which are available in ACA_0, we can define f in ACA_0 and form a sequence of conditions $(F_1, m_1, \delta_1) > (F_2, m_2, \delta_2) > \cdots$ giving $G = \bigcup_k F_k$.

It remains to use $G = \{g_0 < g_1 < \cdots\}$ to complete the proof. By construction, for each non-empty finite subset U of G, there is color $i_U < \ell$ such that $c(x_{U,n}) = i_U$ for all $n \in G$ with $n > \max U$. Define $d : \mathcal{P}_{\text{fin}}(\omega) \to \ell$ by $d(F) = i_{\{g_m : m \in F\}}$. We apply Hindman's theorem to d to obtain an IP set X and a color $i < \ell$. Since X contains an infinite sequence of pairwise disjoint members, we can find a sequence $E_1, E_2, \ldots$ of members of X such that $\max E_k < \min E_{k+1}$. Define $p \in (\omega)^\omega$ to be the partition whose blocks are $p^{-1}(0) = \omega - \bigcup_k \{g_m : m \in E_k\}$ and, for each $k \geq 1$, $p^{-1}(k) = \{g_m : m \in E_k\}$. Note that for all $k \geq 1$,

$$\max p^{-1}(k) = \max\{g_m : m \in E_k\} < \min p^{-1}(k+1) = \min\{g_m : m \in E_{k+1}\}.$$

It remains to verify that p and i have the desired properties. Consider any $x \in (p)^3$; we must show that $c(x) = i$. Let $U = x^{-1}(1) \cap \mu^x(2)$ and let $n = \mu^x(2)$. Then $n = \mu^{x_{U,n}}(2)$ and $x \upharpoonright n = x_{U,n} \upharpoonright n$, so since c is reduced, $c(x) = c(x_{U,n})$. Therefore, it suffices to show $c(x_{U,n}) = i$.

We claim U is a finite union of p-blocks. Because x is a coarsening of p, $x^{-1}(1)$ is a (possibly infinite) union of p-blocks $p^{-1}(j_1) \cup p^{-1}(j_2) \cup \cdots$ with $0 < j_1 < j_2 < \cdots$ and $n = \mu^x(2) = \min x^{-1}(2) = \min p^{-1}(b)$ for some $b \geq 2$. Let j_a be the largest index such that $j_a < b$. Since the p-blocks are finite and increasing, $U = x^{-1}(1) \cap \mu^x(2) = p^{-1}(j_1) \cup \cdots \cup p^{-1}(j_a)$. Note that $n \in G$ (because $p^{-1}(b) \neq p^{-1}(0)$) and $\max U < n$.

It follows that $U = \{g_m \mid m \in F\}$ where $F = E_{j_1} \cup \cdots \cup E_{j_a}$. Since our fixed IP set X is closed under finite unions, $F \in X$ and therefore $d(F) = i$. By the definition of d, $d(F) = i_{\{g_m | m \in F\}} = i_U$, so $i = i_U$. Finally, U is a finite subset of G, $n \in G$ and $\max U < n$, so $c(x_{U,n}) = i_U = i$ as required. □

Observe that this proof of CDRT^3_ℓ from HT produces a homogeneous p with a special property: $\max p^{-1}(i) < \min p^{-1}(i+1)$ for all $i > 0$. We show that this strengthened "ordered finite block" version of CDRT^3_ℓ is equivalent to HT. However, there is no finite block version of CDRT^k_ℓ for $k > 3$.

Proposition 3.14 (RCA_0)**.** *If for every ℓ-coloring of $(< \omega)^2$ there is an infinite homogeneous partition p with $\max p^{-1}(i) < \min p^{-1}(i+1)$ for all $i > 0$, then Hindman's Theorem for ℓ-colorings holds.*

Proof. Given $c : \mathcal{P}_{\text{fin}}(\omega) \to \ell$, define $\widehat{c} : (< \omega)^2 \to \ell$ by $\widehat{c}(\sigma) = c(\{i < |\sigma| : \sigma(i) = 1\})$. Let p be a homogeneous partition for $\widehat{c}$ with $\max p^{-1}(i) < \min p^{-1}(i+1)$ for all $i > 0$. The set of all finite unions of the blocks $p^{-1}(i)$ for $i > 0$ satisfies the conclusion of Hindman's Theorem. □

Proposition 3.15. *There is a 2-coloring of $(< \omega)^3$ such that any infinite homogeneous partition p has $p^{-1}(i)$ infinite for all $i > 0$.*

Proof. For $\sigma \in (< \omega)^3$, set $c(\sigma) = 1$ if σ contains more 1's than 2's and set $c(\sigma) = 0$ otherwise. Let p be homogeneous for this coloring. Suppose for contradiction that $i > 0$ is such that $p^{-1}(i)$ is finite. Let $N = i + 2 + |p^{-1}(i)|$ and let $x = w \circ p$ where

$$w(n) = \begin{cases} 1 & \text{if } n = i \\ 2 & \text{if } i < n \leq N \\ 3 & \text{if } n = N+1 \\ 0 & \text{otherwise} \end{cases}$$

Since x^* has more 2's than 1's, $c(x^*) = 0$. Now coarsen in a different way: let $h \in [i+1, N]$ be chosen so that the size of $p^{-1}(h) \cap [0, \mu^x(3)]$ is minimized. Let

$y = z \circ p$ where

$$z(n) = \begin{cases} 1 & \text{if } i \leq n \leq N \text{ and } n \neq h \\ 2 & \text{if } n = h \\ 3 & \text{if } n = N+1 \\ 0 & \text{otherwise.} \end{cases}$$

Since at least one p-block has moved from $x^{-1}(2)$ to $y^{-1}(1)$ and since $y^{-1}(2)$ contains only the smallest p-block from $x^{-1}(2)$, $c(y^*) = 1$. So p was not homogeneous. □

3.3. CDRT **and the Carlson-Simpson Lemma.** The Carlson-Simpson Lemma is the main technical tool in the original proof of the Borel version of the Dual Ramsey Theorem. The principle is usually stated in the framework of variable words, but it can also be understood as a special case of the Combinatorial Dual Ramsey Theorem.

Carlson-Simpson Lemma ($\mathsf{CSL}(m, \ell)$)**.** *For every coloring $c : (<\omega)^m \to \ell$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $x \in (p)^{m+1}$, if $p^{-1}(j) \subseteq x^{-1}(j)$ for each $j < m$, then $c(x^*) = i$.*

The condition $p^{-1}(j) \subseteq x^{-1}(j)$ for $j < m$ captures those $x \in (p)^{m+1}$ which keep the first m many blocks of p distinct in x. Therefore, $\mathsf{CSL}(m, \ell)$ is a special case of CDRT^{m+1}_ℓ. Two related principles, $\mathsf{OVW}(m, \ell)$ and $\mathsf{VW}(m, \ell)$ have also been studied (see [12, 7, 11]). We do not deal with these principles, but it may be useful to note that $\mathsf{VW}(m, \ell)$ is the strengthening of $\mathsf{CSL}(m, \ell)$ which requires each nonzero block $p^{-1}(j)$ to be finite, and $\mathsf{OVW}(m, \ell)$ is the further strengthening which requires $\max p^{-1}(j) < \min p^{-1}(j+1)$ for all $j > 0$.

In Proposition 3.16, we give three equivalent versions of the Carlson-Simpson Lemma. The version in Proposition 3.16(2) is (up to minor notational changes which are easily translated in RCA_0) the statement from Lemma 2.4 of Carlson and Simpson [4].

Proposition 3.16 (RCA_0)**.** *The following are equivalent.*

1. $\mathsf{CSL}(m, \ell)$.
2. *For each coloring $c : (<\omega)^m \to \ell$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $j < m$, $p(j) = j$ and for all $x \in (p)^{m+1}$, if $p^{-1}(j) \subseteq x^{-1}(j)$ for each $j < m$, then $c(x^*) = i$.*
3. *For each $y \in (\omega)^\omega$ and open reduced coloring $(y)^{m+1} = \cup_{i<\ell} O_i$, there is a partition $p \in (y)^\omega$ and a color i such that for all $j < m$, $y^{-1}(j) \subseteq p^{-1}(j)$ and for all $x \in (p)^{m+1}$, if $p^{-1}(j) \subseteq x^{-1}(j)$ for each $j < m$, then $x \in O_i$.*

Proof. (2) implies (1) because $\mathsf{CSL}(m, \ell)$ is a special case of (2). The extra condition in (2) that $p(j) = j$ for $j < m$ says that the partition p does not collapse any of the first m-many blocks of the trivial partition defined by the identity function. The equivalence between (2) and (3) is proved in a similar way to Proposition 3.6.

It remains to prove (1) implies (2). Fix an ℓ-coloring $c : (<\omega)^m \to \ell$. Define $\tilde{c} : (<\omega)^m \to \ell$ by $\tilde{c}(\sigma) = c(0^\frown 1^\frown \cdots ^\frown (m-1)^\frown \sigma)$. Apply $\mathsf{CSL}(m, \ell)$ to $\tilde{c}$ to get $\tilde{p} \in (\omega)^\omega$ and $i < \ell$ such that for all $\tilde{x} \in (\tilde{p})^{m+1}$, if $\tilde{p}^{-1}(j) \subseteq \tilde{x}^{-1}(j)$ for all $j < m$, then $\tilde{c}(\tilde{x}^*) = i$.

Let $p \in (\omega)^\omega$ be the partition defined by

$$p(j) = \begin{cases} j & \text{if } j < m \\ \tilde{p}(j-m) & \text{if } j \geq m \end{cases}.$$

We claim that p satisfies the conditions in (2) for the coloring c with the fixed color i. Fix $x \in (p)^{m+1}$ such that $p^{-1}(j) \subseteq x^{-1}(j)$ for all $j < m$. We need to show that $c(x^*) = i$. Since x does not collapse any of the first m-many p-blocks, $x(j) = j$ for all $j < m$. Define $\tilde{x} \in (\tilde{p})^{m+1}$ by $\tilde{x}(j) = x(j+m)$. Then $\tilde{p}^{-1}(j) \subseteq \tilde{x}^{-1}(j)$ for all $j < m$. Therefore, $\tilde{c}(\tilde{x}^*) = i$. Now, $x^* = 0^\frown 1^\frown \ldots ^\frown (m-1)^\frown \tilde{x}^*$. Therefore, $c(x^*) = \tilde{c}(\tilde{x}^*) = i$, as required to complete the proof that (1) implies (2). ☐

Let $y \in (\omega)^\omega$ and $(y)^k = \cup_{i<\ell} C_i$ be an m-reduced coloring for some $1 < m < k$. We define the *induced coloring* $(y)^{m+1} = \cup_{i<\ell} \widehat{C}_i$ as follows. For $\widehat{q} \in (y)^{m+1}$, $\widehat{q} \in \widehat{C}_i$ if and only if $q \in C_i$ for some (or equivalently all) $q \in (y)^k$ such that $\widehat{q} \upharpoonright \mu^{\widehat{q}}(m) = q \upharpoonright \mu^q(m)$. This induced coloring is a reduced coloring of $(y)^{m+1}$ and therefore we can apply $\mathsf{CSL}(m, \ell)$ to it.

Our proof of CDRT^k_ℓ from the Carlson-Simpson Lemma will use repeated applications of the following lemma, which is proved using ω many nested applications of $\mathsf{CSL}(m, \ell)$.

Lemma 3.17. *Fix $1 < m < k$ and $y \in (\omega)^\omega$. Let $(y)^k = \cup_{i<\ell} C_i$ be an m-reduced coloring. There is an $x \in (y)^\omega$ such that the coloring restricted to $(x)^k$ is $(m-1)$-reduced.*

Proof. Fix an m-reduced coloring $(y)^k = \cup_{i<\ell} C_i$. We define a sequence of infinite partitions $x_m, x_{m+1}, \cdots$ starting with index m such that $x_m = y$ and x_{s+1} is a coarsening of x_s for which ${x_s}^{-1}(j) \subseteq {x_{s+1}}^{-1}(j)$ for all $j < s$. That is, we do not collapse any of the first s-many blocks of the partition x_s when we coarsen it to x_{s+1}. This property guarantees that the sequence has a well-defined limit $x \in (\omega)^\omega$. We show this limiting partition x satisfies the conclusion of the lemma.

Assume x_s has been defined for a fixed $s \geq m$ and we construct x_{s+1}. Set $x^0_s = x_s$. Let $\sigma_0, \ldots, \sigma_r$ be a list of the elements of $(s)^m$. We define a sequence of coarsenings $x^1_s, \ldots, x^r_s$ and set $x_{s+1} = x^r_s$.

Assume that x^j_s has been defined. Define

$$\sigma^+_j(n) = \begin{cases} \sigma_j(n) & \text{if } n < s \\ m + (n-s) & \text{if } n \geq s \end{cases},$$

and let $w^j_s = \sigma^+_j \circ x^j_s$. That is w^j_s collapses the first s-many blocks of x^j_s into m-many blocks in the j-th possible way and leaves the remaining blocks of x^j_s unchanged. Since w^j_s is a coarsening of y, the coloring $\cup_{i<\ell} C_i$ is also an m-reduced coloring of $(w^j_s)^k$. Let $(w^j_s)^{m+1} = \cup_{i<\ell} \widehat{C}_i$ be the induced coloring. This coloring is reduced, so let $i^j_s < \ell$ and $z^j_s \in (w^j_s)^\omega$ be the result of applying $\mathsf{CSL}(m, l)$ as stated in Proposition 3.16(3). Then z^j_s leaves the first m blocks of w^j_s separate, and any coarsening of z^j_s into at least $m+1$ pieces receives color i^j_s, provided the first m blocks are left separate.

To define x^{j+1}_s, we want to "uncollapse" the first m-many blocks of z^j_s to reverse the action of σ^+_j in defining w^j_s. Since w^j_s collapsed the first s-blocks of x^j_s to m-many blocks and since z^j_s is a coarsening of w^j_s, if $x^j_s(u) < s$, then $z^j_s(u) < m$. We define x^{j+1}_s by cases as follows.

(1) If $x^j_s(u) < s$, then $x^{j+1}_s(u) = x^j_s(u)$.
(2) If $x^j_s(u) \geq s$ and $z^j_s(u) = a < m$, then $x^{j+1}_s(u) = x^j_s(\mu^{z^j_s}(a))$.
(3) If $z^j_s(u) \geq m$, then $x^{j+1}_s(u) = z^j_s(u) + (s - m)$.

Below we verify that x^{j+1}_s is an infinite partition coarsening x^j_s which does not collapse any of the first s-many blocks of x^j_s. This completes the construction of x^{j+1}_s and hence of x_{s+1} and x.

We verify the required properties of x^{j+1}_s. By (1), ${x^j_s}^{-1}(a) \subseteq {x^{j+1}_s}^{-1}(a)$ for all $a < s$, so we do not collapse any of the first s-many blocks of x^j_s in x^{j+1}_s. There is no conflict between (1) and (3) because $x^j_s(u) < s$ implies $z^j_s(u) < m$. Furthermore, (3) renumbers the z^j_s-blocks starting with index m to x^{j+1}_s-blocks starting with index s without changing any of these blocks. Therefore, x^{j+1}_s is an infinite partition.

In (2), we handle the case when the x^j_s-block containing u is not changed by w^j_s (except to renumber its index) but is collapsed by z^j_s into one of the first m-many z^j_s-blocks. In this case, $\mu^{z^j_s}(a) = \mu^{x^j_s}(b)$ for some $b < s$ and we have set $x^{j+1}_s(u) = b$. It is straightforward to check (as in the proof of Theorem 3.8) that x^{j+1}_s is a coarsening of x^j_s and that $\sigma^+_j \circ x^{j+1}_s = z^j_s$.

To complete the proof, we verify that the restriction of $\cup_{i<\ell} C_i$ to $(x)^k$ is $(m-1)$-reduced. Fix $p \in (x)^k$ and we show the color of p depends only on $p \upharpoonright \mu^p(m-1)$.

Let $q \in (\omega)^\omega$ be the unique element with $p = q \circ x$, and let $\sigma = q \upharpoonright \mu^q(m-1)$. Then $\sigma^\frown(m-1) \in (s)^m$ for some s, and

$$p \upharpoonright \mu^p(m-1) = \sigma \circ (x \upharpoonright \mu^x(s-1)).$$

During stage s and afterward, the first s blocks of x_s are always kept separate. Therefore, the above equation remains true when x is replaced with x^{j+1}_s, where j is the unique index such that $\sigma_j = \sigma$. Therefore, p is a coarsening of $\sigma^+_j \circ x^{s+1}_j = z^j_s$ and p keeps the first m blocks of z^j_s separate. Therefore, the color of p is i^j_s, the homogeneous color obtained when we applied $\mathsf{CSL}(m, \ell)$ to obtain z^j_s. This completes the proof that the restriction of $\cup_{i<\ell} C_i$ to $(x)^k$ is $(m-1)$-reduced because the indices s and j in z^j_s are determined only by $p \upharpoonright \mu^p(m-1)$. □

We end this section with the proof of CDRT^k_ℓ.

Theorem 3.18. *For all for $k \geq 2$ and all ℓ, CDRT^k_ℓ holds.*

Proof. For $k = 2$, CDRT^k_ℓ follows from the pigeonhole principle as in Proposition 3.10. Now assume $k \geq 3$. Consider CDRT^k_ℓ in the form given in Proposition 3.6. Let $y \in (\omega)^\omega$ and $(y)^k = \cup_{i<\ell} O_i$ be an open reduced coloring. These satisfy the assumptions of Lemma 3.17 with $m = k-1$. After $k-2$ applications of Lemma 3.17, we obtain $x \in (y)^\omega$ such that the restriction of $\cup_{i<\ell} O_i$ to $(x)^k$ is 1-reduced and hence the color of $p \in (x)^k$ depends only on $p \upharpoonright \mu^p(1)$. Since the numbers $n < \mu^p(1)$ must lie in $p^{-1}(0)$, the color of p is determined by the value of $\mu^p(1)$. By the pigeonhole principle, there is an infinite set $X \subseteq \{\mu^x(a) : a \geq 1\}$ and a color i such that for all $p \in (x)^k$, if $\mu^p(1) \in X$, then $p \in C_i$. It follows that for any $z \in (x)^\omega$ such that $\mu^z(a) \in X$ for all $a \geq 1$, $(z)^k \subseteq C_i$ as required. □

It is interesting to note that the only non-constructive steps in this proof are the $\omega \cdot (k-2)$ nested applications of the Carlson-Simpson Lemma.

4. The Borel Dual Ramsey Theorem for $k \geq 3$

In the next two sections we consider the Borel Dual Ramsey Theorem from the perspective of effective mathematics. We define Borel codes for topologically $\mathbf{\Sigma}^0_\alpha$ subsets of $(\omega)^k$ by induction on the ordinals below ω_1. Let L be some countable set of labels which effectively code for the clopen sets $\emptyset$, $(\omega)^k$ and $[\sigma]$ and $\overline{[\sigma]}$ for $\sigma \in (<\omega)^k$.

Definition 4.1. We define a *Borel code for a* $\mathbf{\Sigma}^0_\alpha$ *or* $\mathbf{\Pi}^0_\alpha$ *set.*

- A *Borel code for a* $\mathbf{\Sigma}^0_0$ *or a* $\mathbf{\Pi}^0_0$ *set* is a labeled tree T consisting of just a root λ in which the root is labeled by a clopen set from L. The Borel code represents that clopen set.
- For $\alpha \geq 1$, a *Borel code for a* $\mathbf{\Sigma}^0_\alpha$ *set* is a labeled tree with a root labeled by $\cup$ and attached subtrees at level 1, each of which is a Borel code for a $\mathbf{\Sigma}^0_{\beta_n}$ or $\mathbf{\Pi}^0_{\beta_n}$ set A_n for some $\beta_n < \alpha$. The code represents the set $\cup_n A_n$.
- For $\alpha \geq 1$, a *Borel code for a* $\mathbf{\Pi}^0_\alpha$ *set* is the same, except the root is labeled $\cap$. The code represents the set $\cap_n A_n$.

For $\alpha \geq 1$, a *Borel code for a* $\mathbf{\Delta}^0_\alpha$ *set* is a pair of labeled trees which encode the same set, where one encodes it as a $\mathbf{\Sigma}^0_\alpha$ set and the other encodes it as a $\mathbf{\Pi}^0_\alpha$ set.

The codes are faithful to the Borel hierarchy in the sense that every code for a $\mathbf{\Sigma}^0_\alpha$ set represents a $\mathbf{\Sigma}^0_\alpha$ set and every $\mathbf{\Sigma}^0_\alpha$ set is represented by a Borel code for a $\mathbf{\Sigma}^0_\alpha$ set. There is a uniform procedure to transform a Borel code B for a $\mathbf{\Sigma}^0_\alpha$ set A into a Borel code $\overline{B}$ for a $\mathbf{\Pi}^0_\alpha$ set $\overline{A}$: leave the underlying tree structure the same, swap the $\cup$ and $\cap$ labels and replace the leaf labels by their complements.

Observe also that a code for a $\mathbf{\Sigma}^0_1$ set essentially agrees with the definition of a code for an open set in Definition 3.1 (up to a primitive recursive translation mapping elements of $\omega \times (<\omega)^k$ to leaves of a $\mathbf{\Sigma}^0_1$ code, and mapping each leaf of a $\mathbf{\Sigma}^0_1$ code to an element or sequence of elements of $\omega \times (<\omega)^k$). The one difference is that we must include a leaf label of $\emptyset$ in the definition of a Borel code, so that the empty set has a $\mathbf{\Sigma}^0_1$ code. Having included $\emptyset$ as a label, we also include $(\omega)^k$ to keep complementation effective.

We recall some notation from hyperarithmetic theory. Let $\mathcal{O}$ denote Kleene's set of computable ordinal notations. The ordinal represented by $a \in \mathcal{O}$ is denoted $|a|_\mathcal{O}$, with $|1|_\mathcal{O} = 0$, $|2^a|_\mathcal{O} = |a|_\mathcal{O} + 1$, and $|3 \cdot 5^e|_\mathcal{O} = \sup_j |\varphi_e(j)|_\mathcal{O}$. The H-sets are defined by effective transfinite recursion on $\mathcal{O}$ as follows: $H_1 = \emptyset$, $H_{2^a} = H'_a$ and $H_{3 \cdot 5^a} = \{\langle i, j\rangle \mid i \in H_{\varphi_a(j)}\}$. The reader is referred to Sacks [16] for more details. To use oracles that line up better than the H-sets do with the levels of the Borel hierarchy, define

$$\emptyset_{(a)} = \begin{cases} H_a & \text{if } |a|_\mathcal{O} < \omega \\ H_{2^a} & \text{otherwise.} \end{cases}$$

If $|a|_\mathcal{O} = |b|_\mathcal{O} = \alpha$, then $\emptyset_{(a)} \equiv_1 \emptyset_{(b)}$, so we sometimes just write $\emptyset_{(\alpha)}$ in that situation. As usual, ω_1^{CK} denotes the least noncomputable ordinal.

Recall the standard effectivizations of the notions described above. We say that a Borel code B is *computable* if it is computable as a labeled subtree of $\omega^{<\omega}$. We say B is *effectively* Σ^0_α (respectively effectively Π^0_α) if the root is labeled $\cup$ (respectively $\cap$) and additionally there is $a \in \mathcal{O}$ with $|a|_\mathcal{O} = \alpha$, and a computable labeling of the nodes of B with notations from $\{b : b \leq_\mathcal{O} a\}$, such that the root is labeled with a and each node has a label strictly greater than all its extensions.

It is well-known that an open set of high hyperarithmetic complexity can be represented by a computable Borel code for a Σ^0_α set, where α is an appropriate computable ordinal. In the following proposition, we use a standard technique to make this correspondence explicit. Fix an effective 1-to-1 enumeration τ_n for the strings $\tau \in (<\omega)^k$.

Proposition 4.2. *There is a partial computable function $p(x, y)$ such that $p(a, e)$ is defined for all $a \in \mathcal{O}$ and $e \in \omega$ and such that if $a \in \mathcal{O}$ and $R = \bigcup\{[\tau_n] : n \in W_e^{\emptyset_{(a)}}\}$, then $\Phi_{p(a,e)}$ is a computable Borel code for R as a $\Sigma^0_{\alpha+1}$ set, where $\alpha = |a|_{\mathcal{O}}$.*

Proof. We define $p(a, e)$ for all e by effective transfinite recursion on $a \in \mathcal{O}$. Let $\Phi_{p(1,e)}$ be a Borel code for the open set $R = \bigcup\{[\tau_n] : n \in W_e\}$.

For the successor step, consider $R = \bigcup\{[\tau_n] : n \in W_e^{\emptyset_{(2^a)}}\}$. Each set which is Σ^0_1 in $\emptyset_{(2^a)}$ is Σ^0_2 in $\emptyset_{(a)}$ and for such sets, we can effectively pass from a $\Sigma^{0,\emptyset_{(2^a)}}_1$ index to a $\Sigma^{0,\emptyset_{(a)}}_2$ description. Specifically, uniformly in e, we compute an index e' such that for all oracles X, $\Phi^X_{e'}(x, y)$ is a total $\{0, 1\}$-valued function and

$$n \in W_e^{X'} \text{ if and only if } \exists t\, \forall s \geq t\, (\Phi^X_{e'}(n, s) = 1).$$

Let $R_t = \bigcup\{[\tau_n] : \exists s \geq t\, (\Phi^{\emptyset_{(a)}}_{e'}(n, s) = 0)\}$. $R_0 \supseteq R_1 \supseteq \cdots$ is a decreasing sequence of sets such that $x \notin R$ if and only if $\forall t\, (x \in R_t)$. Therefore, $R = \cup_t \overline{R_t}$. Each set R_t can be represented as $R_t = \bigcup\{[\tau_n] : n \in W_{e_t}^{\emptyset_{(a)}}\}$, where e_t is uniformly computable from e and t. Applying the induction hypothesis, we define $p(2^a, e)$ to encode a tree whose root is labeled by a union and whose t-th subtree at level 1 is the Borel code representing the complement of $\Phi_{p(a,e_t)}$.

For the limit step, consider $R = \bigcup\{[\tau_n] : n \in W_e^{\emptyset_{(3\cdot 5^d)}}\}$. Uniformly in e, we construct a sequence of indices e_t for $t \in \omega$ such that for all oracles X, $\Phi^X_{e_t}(x)$ converges if and only if $\Phi^X_e(x)$ converges and only asks oracle questions about numbers in the first t many columns of X. Let $R_t = \bigcup\{[\tau_n] : n \in W_{e_t}^{\oplus_{i \leq t} \emptyset_{(\varphi_d(i))}}\}$ and note that $R = \cup_t R_t$. We can effectively pass to a sequence of indices e'_t such that $R_t = \bigcup\{[\tau_n] : n \in W_{e'_t}^{\emptyset_{(\varphi_d(t))}}\}$. By induction, each $p(\varphi_d(t), e'_t)$ is the index for a computable Borel code for R_t as a $\Sigma^0_{2^{\varphi_d(t)}}$ set, so we may define $p(3 \cdot 5^d, e)$ to be the index of a tree which has $\cup$ at the root and $\Phi_{p(\varphi_d(t),e'_t)}$ as its subtrees. Since $2^{\varphi_d(t)} <_{\mathcal{O}} 3 \cdot 5^d$ for all t, the resulting Borel code has the required height. □

To force the Dual Ramsey Theorem to output computationally powerful homogeneous sets, we use the following definition and a result of Jockusch [10].

Definition 4.3. For functions $f, g : \omega \to \omega$, we say g *dominates* f, and write $g \succeq f$, if $f(n) \leq g(n)$ for all but finitely many n.

Theorem 4.4 (Jockusch [10], see also [15, Exercise 16-98])**.** *For each computable ordinal α, there is a function f_α such that $f_\alpha \equiv_T \emptyset_{(\alpha)}$ and for every $g \succeq f_\alpha$, we have $\emptyset_{(\alpha)} \leq_T g$.*

In Theorem 4.7, we use these functions f_α to show that for every computable ordinal α, there is a computable Borel code for a set $R \subseteq (\omega)^3$ such that any homogeneous partition $p \in (\omega)^\omega$ for the coloring $(\omega)^3 = R \cup \overline{R}$ computes $\emptyset_{(\alpha)}$.

Theorem 4.5. *Let A be a set and f_A be a function such that $A \equiv_T f_A$ and for every $g \succeq f_A$, we have $A \leq_T g$. There is an A-computable clopen coloring $(\omega)^3 = R \cup \overline{R}$ for which every homogeneous partition p satisfies $p \geq_T A$.*

Proof. Fix A and f_A as in the statement of the theorem. Without loss of generality, we assume that if $n < m$, then $f_A(n) < f_A(m)$. For $x \in (\omega)^3$, let $a_x = \mu^x(1)$ and $b_x = \mu^x(2)$. Let $O_{a,b} = \{x \in (\omega)^3 : a_x = a \wedge b_x = b\}$. Set $R = \{x \in (\omega)^3 : f_A(a_x) \leq b_x\}$. Since $R = \bigcup\{O_{n,m} \mid f_A(n) \leq m\}$ and $\overline{R} = \bigcup\{O_{n,m} \mid f_A(n) > m\}$ both R and $\overline{R}$ are A-computable open sets.

Claim. If $p \in (\omega)^\omega$ is homogeneous, then $(p)^3 \subseteq R$.

It suffices to show that there is an $x \in (p)^3$ with $x \in R$. Let $u = \mu^p(1)$. Because p has infinitely many blocks, there must be some i with $\mu^p(i) \geq f(u)$. Consider the partition $x = w \circ p$, where $w(1) = 1, w(i) = 2$, and $w(m) = 0$ for all other m. Then since $a_x = u$ and $b_x \geq f(u)$, we have $x \in (p)^3$ with $f(a_x) \leq b_x$, so $x \in R$.

Claim. If $p \in (\omega)^\omega$ is homogeneous, then $A \leq_T p$.

Fix p and let $g(n) = \mu^p(n+2)$. Since g is p-computable, it suffices to show $g \succeq f_A$. Because $n < \mu^p(n+1)$ and f_A is increasing, we have $f_A(n) < f_A(\mu^p(n+1))$. Therefore, to show $g \succeq f_A$, it suffices to show $f_A(\mu^p(n+1)) \leq \mu^p(n+2) = g(n)$.

Let $x_n \in (p)^3$ be defined by $x_n = w_n \circ p$, where $w_n(n+1) = 1, w_n(n+2) = 2$, and $w_n(m) = 0$ for all other m. Note that $a_{x_n} = \mu^p(n+1)$ and $b_{x_n} = \mu^p(n+2)$. By the previous claim, $x_n \in R$, so $f_A(a_{x_n}) \leq b_{x_n}$. In other words, $f_A(\mu^p(n+1)) \leq \mu^p(n+2)$ as required. □

Corollary 4.6. *For each $k \geq 3$ and each recursive ordinal α, there is an $\emptyset_{(\alpha)}$-computable clopen set $R \subseteq (\omega)^k$ such that if $p \in (\omega)^\omega$ is homogeneous for $(\omega)^k = R \cup \overline{R}$, then $\emptyset_{(\alpha)} \leq_T p$.*

Proof. For $k = 3$, this corollary follows from Theorems 4.4 and 4.5. For $k > 3$, use similiar definitions for R and $\overline{R}$ ignoring what happens after the first three blocks of the partition. □

Theorem 4.7. *For every recursive ordinal α, and every $k \geq 3$, there is a computable Borel code for a $\Delta^0_{\alpha+1}$ set $R \subseteq (\omega)^k$ such that every $p \in (\omega)^\omega$ homogeneous for the coloring $(\omega)^k = R \cup \overline{R}$ computes $\emptyset_{(\alpha)}$.*

Proof. Let $R, \overline{R}$ be the $\emptyset_{(\alpha)}$-computable clopen sets from the previous corollary. By Proposition 4.2, both R and $\overline{R}$ have computable Borel codes as $\Sigma^0_{\alpha+1}$ subsets of $(\omega)^k$. Therefore, R has a computable Borel code as $\Delta^0_{\alpha+1}$ set. By the previous corollary, if p is homogeneous for $(\omega)^k = R \cup \overline{R}$, then $p \geq_T \emptyset_{(\alpha)}$, as required. □

For $\alpha = 1$, Theorem 4.7 says there is a Δ^0_2 clopen set $R \subseteq (\omega)^3$ such that R and $\overline{R}$ have computable Borel codes as Σ^0_2 sets (and hence as Δ^0_2 sets) and any homogeneous partition for $(\omega)^3 = R \cup \overline{R}$ computes $\emptyset'$.

5. The Borel Dual Ramsey Theorem for $k = 2$

5.1. **Effective Analysis.** We consider the complexity of finding infinite homogeneous partitions for colorings $(\omega)^2 = R \cup \overline{R}$ as a function of the descriptive complexity of R and/or $\overline{R}$. We begin by showing that if R is a computable open set, there is a computable homogeneous partition.

Theorem 5.1. *Let R be a computable code for an open set in $(\omega)^2$. There is a computable $p \in (\omega)^\omega$ such that $(p)^2 \subseteq R$ or $(p)^2 \subseteq \overline{R}$.*

Proof. If there is an $n \geq 1$ such that $[0^n] \cap R = \emptyset$, then the partition $x \in (\omega)^\omega$ with blocks $\{0, 1, \ldots, n\}, \{n+1\}, \{n+2\}, \ldots$ satisfies $(x)^2 \subseteq \overline{R}$. Otherwise, for arbitrarily large n there are $\tau \succ 0^n 1$ with $[\tau] \subseteq R$, and hence there is a computable sequence $\tau_1, \tau_2, \ldots$ of such τ with $0^i \prec \tau_i$. Computably thin this sequence so that for each i, $0^{|\tau_i|} \prec \tau_{i+1}$. The partition x with blocks $x^{-1}(i) = \{j : \tau_i(j) = 1\}$ for $i > 0$ satisfies $(x)^2 \subseteq R$. □

To extend to sets coded at higher finite levels of the Borel hierarchy, we will need the following generalization of the previous result.

Theorem 5.2. *Let R be a computable code for an open set in $(\omega)^2$ such that $R \cap [0^n] \neq \emptyset$ for all n. Let $\{D_i\}_{i<\omega}$ be a uniform sequence of computable codes for open sets such that each D_i is dense in R. There is a computable $x \in (\omega)^\omega$ such that $(x)^2 \subseteq R \cap (\cap_i D_i)$.*

Proof. We build x as the limit of an effective sequence $\tau_0 \prec \tau_1 \prec \cdots$ with $\tau_s \in (<\omega)^{s+1}$. We define the strings τ_s in stages starting with $\tau_0 = \langle 0 \rangle$ which puts $x(0) = 0$. For $s \geq 1$, we ensure that at the start of stage $s+1$, we have $[\sigma \circ \tau_s] \subseteq R$ for all $\sigma \in (s+1)^2$. That is, the open sets in $(\omega)^2$ determined by each way of coarsening the $s+1$ many blocks of τ_s to two blocks is contained in R.

At stage $s+1$, assume we have defined $\tau_s \in (<\omega)^{s+1}$. If $s \geq 1$, assume that for all $\sigma \in (s+1)^2$, $[\sigma \circ \tau_s] \subseteq R$. Let $\sigma_0, \ldots, \sigma_{M_s - 1}$ list the strings $\sigma \in (s+2)^2$. We define a sequence of strings $\tau_s^0 \prec \cdots \prec \tau_s^{M_s}$ and set $\tau_{s+1} = \tau_s^{M_s}$.

We define τ_s^0 to start a new block as follows. Since $[0^{|\tau_s|}] \cap R \neq \emptyset$, we effectively search for $\gamma_s \in (<\omega)^2$ such that $0^{|\tau_s|} \prec \gamma_s$ and $[\gamma_s] \subseteq R$. Since $\gamma_s \in (<\omega)^2$, there is at least one $m < |\gamma_s|$ such that $\gamma_s(m) = 1$. Define τ_s^0 with $|\tau_s^0| = |\gamma_s|$ by

$$\tau_s^0(m) = \begin{cases} \tau_s(m) & \text{if } m < |\tau_s| \\ s+1 & \text{if } \gamma_s(m) = 1 \text{ (and hence } m \geq |\tau_s|) \\ 0 & \text{if } m \geq |\tau_s| \text{ and } \gamma_s(m) = 0. \end{cases}$$

Note that $\tau_s \prec \tau_s^0$, and that $[\sigma \circ \tau_s^0] \subseteq R$ for all $\sigma \in (s+2)^2$. To see the latter, let j be least such that $\sigma(j) = 1$ and consider two cases. If $j < s+1$, then $\sigma \upharpoonright s+1 \in (s+1)^2$ and the conclusion follows by the induction hypothesis. If $j = s+1$, then $\sigma \circ \tau_s^0 = \gamma_s$.

We continue to define the τ_s^j strings by induction. Assume that τ_s^j has been defined and consider the j-th string σ_j enumerated above describing how to collapse $(s+2)$ many blocks into 2 blocks. Since $\tau_s^0 \preceq \tau_s^j$, we have $\sigma_j \circ \tau_s^0 \preceq \sigma_j \circ \tau_s^j$ and hence $[\sigma_j \circ \tau_s^j] \subseteq R$. Because $\cap_{n<s+1} D_n$ is dense in R, we can effectively search for a string $\delta_s^j \in (<\omega)^2$ such that $\sigma_j \circ \tau_s^j \preceq \delta_s^j$ and $[\delta_s^j] \subseteq \cap_{n<s+1} D_n$. To define τ_s^{j+1}, we uncollapse δ_s^j. Let j^* be the least number such that $\sigma_j(j^*) = 1$. Define

$$\tau_s^{j+1}(m) = \begin{cases} \tau_s^j(m) & \text{if } m < |\tau_s^j| \\ j^* & \text{if } m \geq |\tau_s^j| \text{ and } \delta_s^j(m) = 1 \\ 0 & \text{if } m \geq |\tau_s^j| \text{ and } \delta_s^j(m) = 0 \end{cases}$$

It is straightforward to check that $\tau_s^j \preceq \tau_s^{j+1}$ and that $\sigma_j \circ \tau_s^{j+1} = \delta_s^j$. This completes the construction of the sequence $\tau_s^0 \preceq \cdots \preceq \tau_s^{M_s}$ and of the computable partition x. It remains to show that if $p \in (x)^2$, then $p \in R$ and $p \in \cap_{n \in \omega} D_n$. Fix $p \in (x)^2$ and let $w \in (\omega)^\omega$ be such that $\omega \circ x = p$. Let s_0 be least such that $w(s_0 + 1) = 1$.

Claim. $p \in R$.

Let $\sigma = (0^{s_0})^\frown 1$, so that $\sigma \prec w$. At stage $s_0 + 1$, we defined $\tau^0_{s_0} \prec x$ with the property that $[\sigma \circ \tau^0_{s_0}] \subseteq R$. Since $\sigma \circ \tau^0_s \prec p$, we have $p \in R$.

Claim. $p \in \cap_{n<\omega} D_n$.

Fix $k \in \omega$ and we show $p \in D_k$. Let $s = \max\{k, s_0\}$. Consider the action during stage $s+1$ of the construction. Let $\sigma = w \restriction (s+2)$. Then $\sigma \in (s+2)^2$, so let j be such that $\sigma_j = \sigma$. We defined δ^j_s and τ^{j+1}_s such that $\sigma_j \circ \tau^{j+1}_s = \delta^j_s$ and $[\delta^j_s] \subseteq \cap_{n<s+1} D_n$, so in particular, $[\delta^s_j] \subseteq D_k$. Since $\tau^{j+1}_s \prec x$, we have $\delta^j_s = \sigma_j \circ \tau^{j+1}_s \prec p$, so $p \in D_k$ as required. □

The next proposition is standard, but we present the proof because some details will be relevant to Theorem 5.4. In the proof, we use codes for open sets as in Definition 3.1.

Proposition 5.3. *Let $n \in \omega$ and let $A \subseteq 2^\omega$ be defined by a Σ^0_{n+1} predicate. There are a Δ^0_{n+1} code U for an open set in $(\omega)^2$, a Δ^0_{n+2} code V for an open set in $(\omega)^2$ and a uniformly Δ^0_{n+1} sequence $\langle D_i : i \in \omega\rangle$ of codes for dense open sets such that $U \cup V$ is dense and for all $p \in \cap_{i\in\omega} D_i$, if $p \in U$, then $p \in A$ and if $p \in V$ then $p \notin A$. Furthermore, the Δ^0_{n+1} and Δ^0_{n+2} indices for U, V and $\langle D_i : i \in \omega\rangle$ can be obtained uniformly from a Σ^0_{n+1} index for A.*

Proof. We proceed by induction on n. Throughout this proof, σ, τ, ρ and δ denote elements of $(<\omega)^2$. In addition to the properties stated in the proposition, we ensure that if $\langle m, \sigma\rangle \in U$ (or V) and $\tau \succeq \sigma$, then there is a k such that $\langle k, \tau\rangle \in U$ (or V respectively). Thus, if $U \cap [\sigma] \neq \emptyset$, then there is $\langle k, \tau\rangle \in U$ with $\sigma \preceq \tau$.

For $n = 0$, we have $X \in A \Leftrightarrow \exists k\, \exists m\, P(m, X \restriction k)$ where $P(x, y)$ is a Π^0_0 predicate. Without loss of generality, we assume that if $P(m, X \restriction k)$ holds, then $P(m', Y \restriction k')$ holds for all $k' \geq k$, $m' \geq m$ and $Y \in 2^\omega$ such that $Y \restriction k = X \restriction k$. Let $U = \{\langle n, \sigma\rangle : P(\sigma, n)\}$, $V = \{\langle 0, \sigma\rangle : \forall x\, \forall \tau \succeq \sigma\, (\neg P(\tau, x))\}$ and $D_i = (<\omega)^2$ for $i \in \omega$. It is straightforward to check these codes have the required properties.

For the induction case, let $A \subseteq 2^\omega$ be defined by a Σ^0_{n+2} predicate, so $X \in A \Leftrightarrow \exists k P(X, k)$ where P is a Π^0_{n+1} predicate. For $k \in \omega$, let $A_k = \{X : \neg P(X, k)\}$. Apply the induction hypothesis to A_k to fix indices (uniformly in k) for the Δ^0_{n+1} codes U_k and $\langle D_{i,k} : i \in \omega\rangle$ and for the Δ^0_{n+2} code V_k so that if $p \in \cap_{i\in\omega} D_{i,k}$, then $p \in U_k$ implies $\neg P(k, p)$ and $p \in V_k$ implies $P(k, p)$. Let

$$U = \{\langle\langle k, m\rangle, \sigma\rangle : \langle m, \sigma\rangle \in V_k\} \text{ and}$$
$$V = \{\langle 0, \sigma\rangle : \forall k\, \forall \tau \succeq \sigma\, \exists m\, \exists \rho \succeq \tau\, \langle m, \rho\rangle \in U_k\}.$$

U is a Δ^0_{n+2} code for $\cup_k V_k$, and V is a Δ^0_{n+3} code such that $\langle m, \sigma\rangle \in V$ if and only if every U_k is dense in $[\sigma]$. We claim that $U \cup V$ is dense. Fix σ and assume $U \cap [\sigma] = \emptyset$, so $V_k \cap [\sigma] = \emptyset$ for all k. Since $U_k \cup V_k$ is dense, $U_k \cap [\tau] \neq \emptyset$ for all $\tau \succeq \sigma$ and all k, so $\langle 0, \sigma\rangle \in V$.

For $i = \langle a_i, b_i\rangle$, define $D_i = D_{a_i,b_i} \cap (U_i \cup V_i)$. D_i has a Δ^0_{n+2} code as a dense open set and the index can be uniformly computed from the indices for U_i, V_i and D_{a_i,b_i}. Furthermore, if $p \in \cap_i D_i$ then $p \in \cap_{i,k} D_{i,k}$ and $p \in \cap_k (U_k \cup V_k)$.

Assume that $p \in \cap_i D_i$. First, we show that if $p \in U$, then $p \in A$. Suppose $p \in U = \cup_k V_k$ and fix k such that $p \in V_k$. Since $p \in \cap_i D_{i,k}$ for this fixed k, $p \notin A_k$ by the induction hypothesis. Therefore, $P(k, p)$ holds and hence $p \in A$.

Second, we show that if $p \in V$ then $p \notin A$. Assume $p \in V$ and fix $\langle 0, \sigma \rangle \in V$ such that $\sigma \prec p$. It suffices to show $\neg P(k, p)$ holds for an arbitrary $k \in \omega$. Since $p \in \cap_i D_i$, we have $p \in U_k \cup V_k$ and $p \in \cap_i D_{i,k}$. If $p \in U_k$, then $\neg P(k, p)$ holds by induction and we are done. Therefore, suppose for a contradiction that $p \in V_k$. Fix $\langle 0, \tau \rangle \in V_k$ such that $\sigma \preceq \tau$ and $\tau \prec p$. Since $\langle 0, \sigma \rangle \in V$ and $\sigma \preceq \tau$, there are $\rho \succeq \tau$ and m such that $\langle m, \rho \rangle \in U_k$, and therefore $[\rho] \subseteq U_k \cap V_k$. This containment is the desired contradiction because $q \in [\rho] \cap \cap_i D_{i,k}$ would satisfy $q \in A_k$ and $q \notin A_k$. □

Theorem 5.4. *For every coloring $(\omega)^2 = R \cup \overline{R}$ such that R is a computable code for a Σ^0_{n+2} set, there is either a $\emptyset^{(n)}$-computable $x \in (\omega)^\omega$ which is homogeneous for $\overline{R}$ or a $\emptyset^{(n+1)}$-computable $x \in (\omega)^\omega$ which is homogeneous for R.*

Proof. Fix R and fix a Π^0_{n+1} predicate $P(k, y)$ such that for $y \in (\omega)^2$, $y \in R \Leftrightarrow \exists k\, P(k, y)$. Let U_k, V_k and $\langle D_{i,k} : i \in \omega \rangle$ be the codes from Proposition 5.3 for $R_k = \{y : \neg P(y, k)\}$. Let $U = \cup_k V_k$, $V = \cup\{[\sigma] : \forall k\ U_k \text{ is dense in } [\sigma]\}$ and D_i, $i \in \omega$, be the corresponding codes for R. We split non-uniformly into cases.

Case 1: Assume V is dense in $[0^\ell]$ for some fixed ℓ. We make two observations. First, U is disjoint from $[0^\ell]$. Therefore, each V_k is disjoint from $[0^\ell]$ and hence each U_k is dense in $[0^\ell]$. Second, suppose $y \in (\bigcap_{i,k} D_{i,k}) \cap (\bigcap_k U_k)$. For each k we have $y \in \cap_i D_{i,k}$ and $y \in U_k$, so $\forall k\, \neg P(k, y)$ holds and hence $y \in \overline{R}$.

We apply Theorem 5.2 relativized to $\emptyset^{(n)}$ to the computable open set $O = [0^\ell]$ (which has nonempty intersection with $[0^j]$ for every j) and the $\emptyset^{(n)}$-computable sequence of codes $D_{i,k}$ and U_k for $i, k < \omega$. By the first observation, each coded set in this sequence is dense in O. Therefore, there is a $\emptyset^{(n)}$-computable $x \in (\omega)^\omega$ such that $(x)^2 \subseteq [0^\ell] \cap (\bigcap_{i,k} D_{i,k}) \cap (\bigcap_k U_k)$. By the second observation, $(x)^2 \subseteq \overline{R}$ as required.

Case 2: Assume V is not dense in $[0^m]$ for any m. In this case, since $U \cup V$ is dense, we have $U \cap [0^m] \neq \emptyset$ for all m. We apply Theorem 5.2 relativized to $\emptyset^{(n+1)}$ to the $\emptyset^{(n+1)}$-computable open set U and the $\emptyset^{(n+1)}$-computable sequence of dense sets D_i for $i \in \omega$ to obtain an $\emptyset^{(n+1)}$-computable x with $(x)^2 \subseteq U \cap (\bigcap_i D_i) \subseteq R$ as required. □

We end this section by showing that the non-uniformity in the proof of Theorem 5.1 is necessary.

Theorem 5.5. *For every Turing functional Δ, there are computable codes R_0 and R_1 for complementary open sets in $(\omega)^2$ such that $\Delta^{R_0 \oplus R_1}$ is not an infinite homogeneous partition for the reduced coloring $(\omega)^2 = R_0 \cup R_1$.*

Proof. Fix Δ. We define R_0 and R_1 in stages as $R_{0,s}$ and $R_{1,s}$. Our construction proceeds in a basic module while we wait for $\Delta_s^{R_{0,s} \oplus R_{1,s}}$ to provide appropriate computations. If these computations appear, we immediately diagonalize and complete the construction.

For the basic module at stage s, put $0^{2s+1}1 \in R_{0,s}$ and $0^{2s+2}1 \in R_{1,s}$. Check whether there is a $0 < k < s$ such that $\Delta_s^{R_{0,s} \oplus R_{1,s}}(i) = 0$ for all $i < k$ and $\Delta_s^{R_{0,s} \oplus R_{1,s}}(k) = 1$. If there is no such k, then we proceed to stage $s + 1$ and continue with the basic module.

If there is such a k, then we stop the basic module and fix $i < 2$ such that $0^k1 \in R_{i,s}$. (Since $k < s$, we have already enumerated 0^k1 into one of $R_{0,s}$ or $R_{1,s}$

depending on whether k is even or odd.) We end the construction at this stage and define $R_i = R_{i,s}$ and $R_{1-i} = R_{1-i,s} \cup \{0^t 1 \mid 2s+2 < t\}$.

This completes the construction. It is clear that R_0 and R_1 are computable codes for complementary open sets and $(\omega)^2 = R_0 \cup R_1$ is a reduced coloring. If the construction never finds an appropriate value k, then $\Delta^{R_0 \oplus R_1}$ is not an element of $(\omega)^\omega$ and we are done. Therefore, assume we find an appropriate value k at stage s in the construction. Fix i such that $0^k 1 \in R_{i,s}$ and assume that $p = \Delta^{R_0 \oplus R_1}$ is a element of $(\omega)^\omega$. We show p is not homogeneous by giving elements $q_0, q_1 \in (p)^2$ such that $q_0 \in R_i$ and $q_1 \in R_{1-i}$.

By construction, $0^k 1 \prec p$. Let $q_0 \in (p)^2$ be any coarsening with $0^k 1 \prec q_0$ Then $q_0 \in R_i$ because $[0^k 1] \subseteq R_i$.

On the other hand, since $p \in (\omega)^\omega$, there are infinitely many p-blocks. Let n be least with $\mu^p(n) > 2s+2$. Let $q_1 \in (p)^2$ be any coarsening for which $q_1 \in [0^{\mu^p(n)} 1]$. Since $\mu^p(n) > 2s+2$, we put $0^{\mu^p(n)} 1 \in R_{1-i}$, so $q_1 \in R_{1-i}$ as required. □

5.2. **Strong reductions for reduced colorings.** In this section, we think of $\mathsf{Borel\text{-}DRT}^2_2$ as an *instance-solution problem.* Such a problem consists of a collection of subsets of ω called the *instances* of this problem, and for each instance, a collection of subsets of ω called the *solutions* to this instance (for this problem). A problem P is *strongly Weihrauch reducible* to a problem Q if there are fixed Turing functionals Φ and Ψ such that given any instance A of P, Φ^A is an instance of Q, and given any solution B to Φ^A in Q, Ψ^B is a solution to A in P. There are a number of variations on this reducibility and we refer to the reader to [6] and [9] for background on these reductions and for connections to reverse mathematics. In this paper, we will only be interested in problems arising out of Π^1_2 statements of second order arithmetic. Any such statement can be put in the form $\forall X(\varphi(X) \rightarrow \exists Y \psi(X,Y))$, where ϕ and ψ are arithmetical. We can then regard this as a problem, with instances being all X such that $\varphi(X)$, and the solutions to X being all Y such that $\psi(X,Y)$. Note that while the choice of φ and ψ is not unique, we always have a fixed such choice in mind for a given Π^1_2 statement, and so also a fixed assignment of instances and solutions.

A reduced coloring $(\omega)^2 = R_0 \cup R_1$ is classically open and the color of $p \in (\omega)^2$ depends only on $\mu^p(1)$. When R_0 and R_1 are codes for open sets, there is a homogeneous partition computable in $R_0 \oplus R_1$, although by Theorem 5.5, not uniformly. We consider the case when the open sets R_0 and R_1 are represented by Borel codes for Σ^0_n sets with $n \geq 2$.

Δ^0_n-rDRT^2_2 is the statement that for each reduced coloring $(\omega)^2 = R_0 \cup R_1$ where R_0 and R_1 are Borel codes for Σ^0_n sets, there exists an $x \in (\omega)^\omega$ and an $i < 2$ such that $(x)^2 \subseteq R_i$. In effective algebra, this statement is clear, but in RCA_0, we need to specify how to handle these codes.

Recall that a Borel code for a $\boldsymbol{\Sigma}^0_n$ set is a labeled subtree of $\omega^{<n+1}$ which we write as (B, φ) to specify the labeling function φ. For a leaf σ and a partition p, we write $p \in \varphi(\sigma)$ if p is an element of the clopen set coded by $\varphi(\sigma)$, and we write $\varphi(\sigma) = [\tau]$ to avoid specifying a coding scheme.

In reverse mathematics there are two ways that membership in a $\boldsymbol{\Sigma}^0_\alpha$ set could be discussed. The *evaluation map* method works for arbitrary α and requires a strong base theory. This method will be discussed in the next section. The *virtual* method works only for finite α. For each $n < \omega$, there is a fixed Σ^0_n formula $\eta(B, \varphi, p)$ such that if (B, φ) is a Borel code for a $\mathbf{\Sigma}^0_n$ set and $p \in (\omega)^2$, then $\eta(B, \varphi, p)$ says p is in the set coded by (B, φ). In this section we use only the virtual method.

The formula is defined as follows. We begin by defining formulas $\beta_k(\sigma, B, \varphi, p)$ for $1 \leq k \leq n$ by downward induction on k. For $\sigma \in B$ with $|\sigma| = k$, $\beta_k(\sigma, B, \varphi, p)$ says that p is in the set coded by the labeled subtree of (B, φ) above σ. Since any $\sigma \in B$ with $|\sigma| = n$ is a leaf, $\beta_n(\sigma, B, \varphi, p)$ is the formula $p \in \varphi(\sigma)$. For $1 \leq k < n$, $\beta_k(\sigma, B, \varphi, p)$ is the formula

$$(\varphi(\sigma) = \cup \to \alpha_k^{\cup}) \wedge (\varphi(\sigma) = \cap \to \alpha_k^{\cap}) \wedge (\varphi(\sigma) \in L \to \alpha_k^{L}), \text{ where}$$
$$\alpha_k^{\cup}(\sigma, B, \varphi, p) \text{ is } \exists \tau \in B\big(\sigma \prec \tau \wedge |\tau| = k+1 \wedge \beta_{k+1}(\tau, B, \varphi, p)\big)$$
$$\alpha_k^{\cap}(\sigma, B, \varphi, p) \text{ is } \forall \tau \in B\big((\sigma \prec \tau \wedge |\tau| = k+1) \to \beta_{k+1}(\tau, B, \varphi, p)\big)$$
$$\text{and } \alpha_k^{L}(\sigma, B, \varphi, p) \text{ is } p \in \varphi(\sigma).$$

The formula $\eta(B, \varphi, p)$ is $\exists \sigma \in B\,(|\sigma| = 1 \wedge \beta_1(\sigma, B, \varphi, p))$. In RCA_0, we write $p \in B$ for $\eta(B, \varphi, p)$. The statement Δ^0_n-rDRT^2_2 now has the obvious translation in RCA_0.

A Borel code (B, φ) for a $\mathbf{\Sigma}^0_n$ set is *in normal form* if $B = \omega^{<n+1}$ and for every σ with $|\sigma| < n$, if $|\sigma|$ is even, then $\varphi(\sigma) = \cup$, and if $|\sigma|$ is odd, then $\varphi(\sigma) = \cap$. In RCA_0, for every (B, φ), there is a $(\widehat{B}, \widehat{\varphi})$ in normal form such that for all $p \in (\omega)^2$, $p \in B$ if and only if $p \in \widehat{B}$. Moreover, the transformation from (B, φ) to $(\widehat{B}, \widehat{\varphi})$ is uniformly computable in (B, φ). We describe the transformation when (B, φ) is a Borel code for a Σ^0_2 set. The case for a Σ^0_n set is similar.

Let (B, φ) be a Borel code for a Σ^0_2 set. By definition, $\lambda \in B$ with $\varphi(\lambda) = \cup$. Each $\sigma \in B$ with $|\sigma| = 1$ is the root of a subtree coding a Σ^0_0 set (if $\varphi(\sigma) \in L$), a Σ^0_1 set (if $\varphi(\sigma) = \cup$) or a Π^0_1 set (if $\varphi(\sigma) = \cap$). Consider the following sequence of transformations.

- To form (B_1, φ_1), for each $\sigma \in B$ with $|\sigma| = 1$ and $\varphi(\sigma) = \cup$, remove the subtree of B above σ (including σ). For each $\tau \in B$ with $\tau \succ \sigma$, add a new node τ' to B_1 with $|\tau'| = 1$ and $\varphi_1(\tau') = \varphi(\tau) \in L$.
- To form (B_2, φ_2), for each leaf $\sigma \in B_1$ with $|\sigma| = 1$, relabel σ by $\varphi_2(\sigma) = \cap$ and add a new successor τ to σ with label $\varphi_2(\tau) = \varphi_1(\sigma)$.
- To form (B_3, φ_3), for each $\sigma \in B_2$ with $|\sigma| = 1$, let $\tau_\sigma \in B_1$ be the first successor of σ. Add infinite many new nodes $\delta \succ \sigma$ to B_3 with $\varphi_3(\delta) = \varphi_2(\tau_\sigma)$.
- To form (B_4, φ_4), let σ be the first node of B_3 at level 1. Add infinitely many copies of the subtree above σ to B_4 with the same labels as in B_3.

In (B_4, φ_4), the leaves are at level 2, every interior node is infinitely branching and $\varphi_4(\sigma) = \cap$ when $|\sigma| = 1$. There is a uniform procedure to define a bijection $f : B_4 \to \omega^{<3}$. We define $(\widehat{B}, \widehat{\varphi})$ by $\widehat{B} = \omega^{<3}$ and $\widehat{\varphi}(\sigma) = \varphi_4(f^{-1}(\sigma))$. In RCA_0, for all $p \in (\omega)^2$, $\eta(B, \varphi, p)$ holds if and only if $\eta(\widehat{B}, \widehat{\varphi}, p)$ holds.

When (B, φ) is a Borel code for a Σ^0_n set in normal form, $\eta(B, \varphi, p)$ is equivalent to $\exists x_0\, \forall x_1 \cdots \mathsf{Q}_{n-1} x_{n-1}\, (p \in \varphi(\langle x_0, x_1, \ldots, x_{n-1}\rangle))$ where Q_{n-1} is $\forall$ or $\exists$ depending on whether $n-1$ is odd or even. We have analogous definitions for Borel codes for Π^0_n sets in normal form.

To define D^n_2, let $[\omega]^n$ denote the set of n element subsets of ω. We view the elements of $[\omega]^n$ as strictly increasing sequences $s_0 < s_1 < \cdots < s_{n-1}$.

Definition 5.6. A coloring $c : [\omega]^n \to 2$ is *stable* if for all k, the limit

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1})$$

exists. $L \subseteq \omega$ is *limit-homogeneous* for a stable coloring c if there is an $i < 2$ such that for each $k \in L$,

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = i.$$

D^n_2 is the statement that each stable coloring $c : [\omega]^n \to 2$ has an infinite limit-homogeneous set.

Below, the proof of Theorem 5.7(2) is a formalization of the proof of Theorem 5.7(1), and the additional induction used is a consequence of this formalization. We do not know if its use is necessary; that is, we do now if $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$ can be replaced simply by RCA_0 when $n > 2$.

Theorem 5.7. *Fix $n \geq 2$.*

(1) Δ^0_n-$\mathsf{rDRT}^2_2 \equiv_{\mathrm{sW}} \mathsf{D}^n_2$.
(2) *Over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$, Δ^0_n-rDRT^2_2 *is equivalent to* D^n_2.

Corollary 5.8. Δ^0_2-rDRT^2_2 *is equivalent to* SRT^2_2 *over* RCA_0.

Proof. D^2_2 is equivalent to SRT^2_2 over RCA_0 by Chong, Lempp, and Yang [5]. □

Corollary 5.9. Δ^0_2-$\mathsf{rDRT}^2_2 <_{\mathrm{sW}} \mathsf{SRT}^2_2$.

Proof. $\mathsf{D}^2_2 <_{\mathrm{sW}} \mathsf{SRT}^2_2$ by Dzhafarov [6, Corollary 3.3]. (It also follows immediately that Δ^0_2-$\mathsf{rDRT}^2_2 \equiv_{\mathrm{W}} \mathsf{D}^2_2 <_{\mathrm{W}} \mathsf{SRT}^2_2$.) □

Proof of Theorem 5.7. We prove the two parts simultaneously, remarking, where needed, how to formalize the argument in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$.

To show that Δ^0_n-$\mathsf{rDRT}^2_2 \leq_{\mathrm{sW}} \mathsf{D}^n_2$, and that Δ^0_n-rDRT^2_2 is implied by D^n_2 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$, fix an instance $(\omega)^2 = R_0 \cup R_1$ of Δ^0_n-rDRT^2_2 where each R_i is a Borel code for a Σ^0_n set. Without loss of generality, R_0 and R_1 are in normal form. For each $k \geq 1$, fix the partition $p_k = \chi_{\{k\}}$ (that is, p_k has blocks $\omega \setminus \{k\}$ and $\{k\}$).

For $m < n$, we let $R_i(t_0, \ldots, t_m)$ denote the Borel set coded by the subtree of R_i above $\langle t_0, \ldots, t_m\rangle$. Since $\langle t_0, \ldots, t_{n-1}\rangle$ is a leaf, $R_i(t_0, \ldots, t_{n-1})$ is the clopen set $\varphi_i(\langle t_0, \ldots, t_{n-1}\rangle)$. If $m < n-1$, then $R_i(t_0, \ldots, t_m)$ is a code for a $\Sigma^0_{n-(m+1)}$ set (if m is odd) or a $\Pi^0_{n-(m+1)}$ set (if m is even) in normal form.

We define a coloring $c : [\omega]^n \to 2$ as follows. Let $c(0, s_1, \ldots, s_{n-1}) = 0$ for all $s_1 < \cdots < s_{n-1}$. For $m \leq n$, let Q_m stand for $\exists$ or $\forall$, depending as m is even or odd, respectively. Given $1 \leq k < s_1 < \ldots < s_{n-1}$, define

$$c(k, s_1, \ldots, s_{n-1}) = 1$$

if and only if there is a $t_0 \leq s_1$ such that

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m) \cdots (\mathsf{Q}_{n-1} t_{n-1} \leq s_{n-1})\ p_k \in \varphi_0(\langle t_0, \ldots, t_{n-1}\rangle)$$

and for which there is no $u_0 < t_0$ such that

$$(\forall u_1 \leq s_1) \cdots (\mathsf{Q}_m u_m \leq s_m) \cdots (\mathsf{Q}_{n-1} u_{n-1} \leq s_{n-1})\ p_k \in \varphi_1(\langle u_0, \ldots u_{n-1}\rangle).$$

(Note that s_1 bounds t_0, t_1 and u_1, whereas the other s_m bound only t_m and u_m.) The coloring c is uniformly computable in (R_0, φ_0) and (R_1, φ_1) and is definable in RCA_0 as a total function since all the quantification is bounded.

We claim that for each $k \geq 1$,

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1})$$

exists. Furthermore, if this limit equals 1, then $p_k \in R_0$, and if this limit equals 0, then $p_k \in R_1$. We break this claim into two halves.

First, for $1 \leq m \leq n-1$, we claim that for all fixed $1 \leq k < s_1 < \ldots < s_m$,

$$\lim_{s_{m+1}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_m, s_{m+1}, \ldots, s_{n-1})$$

exists, and the limit equals 1 if and only if there is a $t_0 \leq s_1$ such that

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m)\ p_k \in R_0(t_0, \ldots, t_m) \tag{2}$$

and there is no $u_0 < t_0$ such that

$$(\forall u_1 \leq s_1) \cdots (\mathsf{Q}_m u_m \leq s_m)\ p_k \in R_1(u_0, \ldots, u_m). \tag{3}$$

The proof is by downward induction on m. (In RCA_0, the induction is performed externally, so we do not need to consider its complexity.) For $m = n-1$, there are no limits involved and the values of c are correct by definition.

Assume the result is true for $m+1$ and we show it remains true for m. By the definition of $R_0(t_0, \ldots, t_m)$, t_0 satisfies (2) if and only if

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m)(\mathsf{Q}_{m+1} t_{m+1})\ p_k \in R_0(t_0, \ldots, t_m, t_{m+1}),$$

which in turn holds if and only if there is a bound v such that for all $s_{m+1} \geq v$,

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m)(\mathsf{Q}_{m+1} t_{m+1} \leq s_{m+1})\ p_k \in R_0(t_0, \ldots, t_m, t_{m+1}).$$

If Q_{m+1} is $\exists$, then over RCA_0, this equivalence requires a bounding principle. Since $p_k \in R_0(t_0, \ldots, t_{m+1})$ is a $\Pi^0_{n-(m+2)}$ predicate and $m+2 \geq 3$, we need at most $\mathsf{B}\Pi^0_{n-3}$ which follows from $\mathsf{I}\Sigma^0_{n-1}$. An analogous analysis applies to numbers u_0 satisfying (3). Thus, we can fix a common bound v that works for all $t_0 \leq s_1$ in (2) and all $u_0 < t_0 \leq s_1$ in (3).

Suppose there is a $t_0 \leq s_1$ satisfying (2) for which there is no $u_0 < t_0$ satisfying (3). Then, for all $s_{m+1} \geq v$, t_0 satisfies the version of (2) for $m+1$, and there is no $u_0 < t_0$ satisfying the version of (3) for $m+1$. Therefore, by induction

$$\exists v \forall s_{m+1} \geq v \big(\lim_{s_{m+2}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 1 \big)$$

and hence $\lim_{s_{m+1}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 1$ as required.

On the other hand, suppose that there is no $t_0 \leq s_1$ satisfying (2), or that for every $t_0 \leq s_1$ satisfying (2), there is a $u_0 < t_0$ satsifying (3). Then, for all $s_{m+1} \geq v$, we have the analogous condition for $m+1$ and the induction hypothesis gives $\lim_{s_{m+1}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 0$. This completes the first part of the claim.

We can now prove the rest of the claim. For each $k \geq 1$, we have $p_k \in R_0$ or $p_k \in R_1$. Let t_0 be least such that $p_k \in R_0(t_0)$ or $p_k \in R_1(t_0)$. Since $p_k \in R_i(t)$ is a Π^0_{n-1} statement, we use $\mathsf{I}\Sigma^0_{n-1}$ to fix this value in RCA_0.

Suppose $p_k \in R_0(t_0)$, so for all $u_0 < t_0$, it is not the case that $p_k \in R_1(u_0)$. By the first half of the claim with $m = 1$, we have for every $s_1 \geq t_0$

$$\lim_{s_2} \cdots \lim_{s_{n-1}} c(k, s_1, s_2, \ldots, s_{n-1}) = 1,$$

and therefore $\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 1$.

Suppose $p_k \notin R_0(t_0)$, and hence $p_k \in R_1(t_0)$. Again, by the first half of the claim with $m = 1$, we have for every $s_1 \geq t_0$

$$\lim_{s_2} \cdots \lim_{s_{n-1}} c(k, s_1, s_2, \ldots, s_{n-1}) = 0,$$

so $\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 0$. This completes the proof of the claim.

Since c is an instance of D^n_2, fix $i < 2$ and an infinite limit-homogeneous set L for c with color i. By the claim, $p_k \in R_{1-i}$ for all $k \in L$. List the non-zero elements of L as $k_0 < k_1 < \cdots$, and let $p \in (\omega)^\omega$ be the partition whose blocks are $[0, k_0)$ and $[k_m, k_{m+1})$ for $m \in \omega$. Each $x \in (p)^2$ satisfies $\mu^x(1) = k_m$ for some m. Since $R_0 \cup R_1$ is a reduced coloring, x and p_{k_m} have the same color, which is R_{1-i}. Since x was arbitrary, $(p)^2 \subseteq R_{1-i}$ as required to complete this half of the theorem.

Next, we show that $\mathsf{D}^n_2 \leq_{\mathrm{sW}} \Delta^0_n\text{-}\mathsf{rDRT}^2_2$, and that D^n_2 is implied by $\Delta^0_n\text{-}\mathsf{rDRT}^2_2$ over RCA_0. (No extra induction is necessary for this implication.) Fix an instance $c : [\omega]^n \to 2$ of D^n_2, and define a partition $R_0 \cup R_1$ of $(\omega)^2$ as follows. For $x \in (\omega)^2$ with $\mu^x(1) = k$, $x \in R_i$ for the unique i such that

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = i.$$

Since each of the iterated limits is assumed to exist over what follows on the right, we may express these limits by alternating Σ^0_2 and Π^0_2 definitions, as

$$(\exists t_1 \forall s_1 \geq t_1)(\forall t_2 \geq s_1 \exists s_2 \geq t_2) \cdots c(k, s_1, \ldots, s_{n-1}) = i.$$

Thus, R_0 and R_1 are Σ^0_n-definable open subsets of $(\omega)^2$. By standard techniques, there are Borel codes for R_0 and R_1 as Σ^0_n sets uniformly computable in c and in RCA_0. (Below, we illustrate this process for D^3_2.)

By definition, $(\omega)^2 = R_0 \cup R_1$ is a reduced coloring and hence is an instance of Δ^0_n-rDRT^2_2. Let $p \in (\omega)^\omega$ be a solution to this instance, say with color $i < 2$. Thus, for every $x \in (p)^2$, the limit color of $k = \mu^x(1)$ is i. Define $L = \{\mu^p(m) : m \geq 1\}$. Since for each $k \in L$, there is an $x \in (p)^2$ such that $\mu^x(1) = k$, L is limit-homogeneous for c with color i.

We end this proof by illustrating how to define the Borel codes for R_0 and R_1 as Σ^0_3 sets from a stable coloring $c(k, s_1, s_2)$. In this case, we have

$$\lim_{s_1} \lim_{s_2} c(k, s_1, s_2) = i \Leftrightarrow \exists t_1 (\forall s_1 \geq t_1\, \forall t_2 \geq s_1)(\exists s_2 \geq t_2)\; c(k, s_1, s_2) = i.$$

The nodes in each R_i are the initial segments of the strings $\langle \langle k, t_1 \rangle, \langle s_1, t_2 \rangle, s_2 \rangle$ for $k \leq t_1 < s_1 \leq t_2 < s_2$ and the labeling functions are $\varphi_i(\sigma) = \cup$ if $|\sigma| \in \{0, 2\}$, $\varphi_i(\sigma) = \cap$ if $|\sigma| = 1$ and $\varphi_i(\langle \langle k, t_1 \rangle, \langle s_1, t_2 \rangle, s_2 \rangle) = [0^k 1]$ if $c(k, s_1, s_2) = i$ and is equal to $\emptyset$ if $c(k, s_1, s_2) = 1 - i$. It is straightforward to check in RCA_0 that R_i represents the union of clopen sets $[0^k 1]$ such that the limit color of k is i. □

6. Reverse math and Borel codes

6.1. **Equivalence of the Borel and Baire versions over ATR_0.** In this subsection we show that over the base theory ATR_0, the Baire and Borel versions of the Dual Ramsey Theorem are equivalent. We make the following definition in reverse mathematics.

Definition 6.1 (RCA_0)**.** A *Borel code* is a pair (B, φ), where $B \subseteq \omega^{<\omega}$ is well-founded and φ is a labeling function as in Definition 4.1.

This definition differs slightly from the definition of a Borel code which is found in the standard reference [17]. In that treatment, there is no labeling function, but certain conventions on the strings in B determine the labels. Because there is no labeling function, the set of leaves of B may not be guaranteed to exist in weak theories. In [17], the base theory for anything to do with Borel sets is ATR_0, so this distinction is never used. We would like to consider weaker base theories. When the base theory is weaker, a constructive presentation of a Borel code should include knowledge of which nodes are leaves. For example, this leaf-knowledge was used in the proof of Theorem 5.7. This is the reason for including the labeling function in our definition.

In Section 5.2 we diverged from the standard definition in a second way, by ascertaining membership in a $\mathbf{\Sigma}^0_n$ set *virtually*. The standard method, which we use in this section, is via evaluation maps.

Definition 6.2 (RCA_0)**.** Let (B, φ) be a Borel code and $x \in (\omega)^k$. An *evaluation map* for B at x is a function $f : B \to \{0, 1\}$ such that

- For leaves $\sigma \in B$, $f(\sigma) = 1$ if and only if $x \in \varphi(\sigma)$.
- If $\varphi(\sigma) = \cup$, $f(\sigma) = 1$ if and only if there exists n such that $\sigma^\frown n \in B$ and $f(\sigma^\frown n) = 1$.
- If $\varphi(\sigma) = \cap$, $f(\sigma) = 1$ if and only if for all n such that $\sigma^\frown n \in B$, $f(\sigma^\frown n) = 1$.

We say $x \in B$ if there is an evaluation map with value 1 at the root, and we say $x \notin B$ if there is an evaluation map with value 0 at the root.

Observe that both $x \in B$ and $x \notin B$ are Σ^1_1 statements. In general, ATR_0 is required to show that evaluation maps exist. Similarly, $(\omega)^k = C_0 \cup \ldots \cup C_{\ell-1}$ is the Π^1_2 statement that for every $x \in (\omega)^k$ and $i < \ell$, there is an evaluation map for C_i at x and for some $i < \ell$, $x \in C_i$.

Definition 6.3 (RCA_0)**.** Let B be a Borel (or open or closed) code for subset of $(\omega)^k$. A *Baire code for B* consists of open sets U and V and a sequence $\langle D_n : n \in \omega\rangle$ of dense open sets such that $U \cup V$ is dense and for every $p \in \cap_{n\in\omega} D_n$, if $p \in U$ then $p \in B$ and if $p \in V$ then $p \notin B$.

Definition 6.4 (RCA_0)**.** A *Baire code* for a Borel coloring $(\omega)^k = C_0 \cup \cdots \cup C_{\ell-1}$ consists of open sets O_i, $i < \ell$, and a sequence $\langle D_n : n \in \omega\rangle$ of dense open sets such that $\cup_{i<\ell} O_i$ is dense and for every $p \in \cap_{n\in\omega} D_n$ and $i < \ell$, if $p \in O_i$ then $p \in C_i$.

We confirm that ATR_0 proves that every Borel set has the property of Baire. This is just an effectivization of the usual proof.

Proposition 6.5 (ATR_0)**.** *Every Borel code for a subset of $(\omega)^k$ has a Baire code.*

Proof. Fix a Borel code B. For $\sigma \in B$, let $B_\sigma = \{\tau \in B : \tau \text{ is comparable to } \sigma\}$. B_σ is a Borel code for the set coded coded by the subtree of B above σ in the following sense. Let f be an evaluation map for B at x. The function $g : B_\sigma \to 2$ defined by $g(\tau) = f(\tau)$ for $\tau \succeq \sigma$ and $g(\tau) = f(\sigma)$ for $\tau \prec \sigma$ is an evaluation map for B_σ at x which witnesses $x \in B_\sigma$ if and only if $f(\sigma) = 1$. We denote this function g by $f_{\sigma,x}$.

Formally, our proof proceeds in two steps. First, by arithmetic transfinite recursion on the Kleene-Brouwer order $KB(B)$, we construct open sets U_σ, V_σ and $D_{n,\sigma}$, $n \in \omega$, which are intended to form a Baire code for B_σ. This construction is essentially identical to the proof of Proposition 5.3. Second, for any $x \in (\omega)^k$ and evaluation map f for B at x, we show by arithmetic transfinite induction on $KB(B)$ that if $x \in \cap_{n\in\omega} D_{n,\sigma}$, then $x \in U_\sigma$ implies $x \in B_\sigma$ via $f_{\sigma,x}$ and $x \in V_\sigma$ implies $x \notin B_\sigma$ via $f_{\sigma,x}$. For ease of presentation, we combine these two steps.

Since ATR_0 suffices to construct evaluation maps, we treat Borel codes as sets in a naive manner and suppress explicit mention of the evaluation maps.

If σ is a leaf coding a basic clopen set $[\tau]$, we set $U_\sigma = [\tau]$, $V_\sigma = \overline{[\tau]}$ and $D_{n,\sigma} = (\omega)^k$. Similarly, if σ codes $\overline{[\tau]}$, we switch the values of U_σ and V_σ. In either case, it is clear that these open sets form a Baire code for B_σ.

Suppose σ is an internal node coding a union, so B_σ is the union of $B_{\sigma^\frown k}$ for $\sigma^\frown k \in B$. We define U_σ to be the union of $U_{\sigma^\frown k}$ for $\sigma^\frown k \in B$ and V_σ to be the union of $[\tau]$ such that $V_{\sigma^\frown k}$ is dense in $[\tau]$ for all $\sigma^\frown k \in B$. The sequence $D_{n,\sigma}$ is the sequence of all open sets of the form $D_{n,\sigma^\frown k} \cap (U_{\sigma^\frown k} \cup V_{\sigma^\frown k})$ for $n \in \omega$ and $\sigma^\frown k \in B$. As in the proof of Proposition 5.3, $U_\sigma \cup V_\sigma$ and each $D_{n,\sigma}$ are dense.

Let $x \in \cap_{n\in\omega} D_{n,\sigma}$. Suppose $x \in U_\sigma$ and we show $x \in B_\sigma$. By the definition of U_σ, fix $\sigma^\frown k \in B$ such that $x \in U_{\sigma^\frown k}$. Since $x \in \cap_{n\in\omega} D_{n,\sigma^\frown k}$, we have by induction that $x \in B_{\sigma^\frown k}$ and hence $x \in B_\sigma$. On the other hand, suppose $x \in V_\sigma$ and we show $x \notin B_\sigma$. Fix τ such that $\tau \prec x$ and $[\tau] \subseteq V_\sigma$, and fix k such that $\sigma^\frown k \in B$. Since $x \in \cap_{n\in\omega} D_{n,\sigma}$, $x \in U_{\sigma^\frown k} \cup V_{\sigma^\frown k}$. However, $V_{\sigma^\frown k}$ is dense in $[\tau]$. Therefore, $x \notin U_{\sigma^\frown k}$ (because $U_{\sigma^\frown k}$ and $V_{\sigma^\frown k}$ must be disjoint as in the proof of Proposition 5.3), so $x \in V_{\sigma^\frown k}$. Since $x \in \cap_{n\in\omega} D_{n,\sigma^\frown k}$, we have by induction that $x \notin B_{\sigma^\frown k}$. Because this holds for every $\sigma^\frown k \in B$, it follows that $x \notin B_\sigma$, completing the case for unions.

The case for an interior node coding an intersection is similar with the roles of U_σ and V_σ switched. □

Proposition 6.6 (ATR_0)**.** $\mathsf{Baire\text{-}DRT}^k_\ell$ *implies* $\mathsf{Borel\text{-}DRT}^k_\ell$.

Proof. By Proposition 6.5, fix Baire codes U_i, V_i and $D_{n,i}$ for each C_i. We claim that the open sets U_i for $i < \ell$ and the sequence of dense open sets $D_{n,i}$ for $i < \ell$ and $n < \omega$ form a Baire code for this coloring. Note that if $i < \ell$ and $x \in \cap_{n,i} D_{n,i}$, then $x \in U_i$ implies $x \in C_i$. Therefore, it suffices to show that $\cup_{i<\ell} U_i$ is dense.

Suppose not. Then there is τ such that $[\tau] \cap U_i = \emptyset$ for all i. Because each set $U_i \cup V_i$ is open and dense, by the Baire Category Theorem there is $x \in [\tau]$ such that $x \in \cap_{n\in\omega, i<\ell} D_{n,i}$ and $x \in \cap_{i<\ell}(U_i \cup V_i)$. Since x is not in any U_i, we have $x \in V_i$ for each i. Therefore, for each i, $x \notin C_i$, contradicting that $(\omega)^k = C_0 \cup \cdots \cup C_{\ell-1}$. □

Lemma 6.7 (RCA_0)**.** *For every code O for an open set, there is a Borel code B such that $(\omega)^k = B \cup \overline{B}$ and for all $x \in (\omega)^k$, $x \in B$ if and only if $x \in O$.*

Proof. The content here lies in the proof that $(\omega)^k = B \cup \overline{B}$. That is, we need to show that in the obvious Borel code, every $x \in (\omega)^k$ has an evaluation map.

Fix O. Let (B, φ) be the Borel code consisting of a root and a single leaf for each $\langle s, \tau\rangle \in O$, where the leaf is labeled with $[\tau]$.

We claim that for every $x \in (\omega)^k$, there is a unique evaluation map f for B at x, and $f(\lambda) = 1$ if and only if $x \in O$. To prove this claim, we define two potential evaluation maps, f_0 and f_1. Let $f_0(\lambda) = 0$ and $f_1(\lambda) = 1$. Then for each $i \in \{0, 1\}$ and each leaf σ with label τ, define $f_i(\sigma) = 1$ if and only if $x \in [\tau]$. Both these functions have $\Delta^0_1(x, B, \varphi)$ definitions, and exactly one of them satisfies the condition to be an evaluation map. Clearly, this condition implies that $x \in B$ if and only if $x \in O$. □

Corollary 6.8 (RCA_0)**.** $\mathsf{Borel\text{-}DRT}^k_\ell$ *implies* $\mathsf{Baire\text{-}DRT}^k_\ell$.

Proof. The previous proposition shows that $\mathsf{Borel\text{-}DRT}^k_\ell$ implies ODRT^k_ℓ and hence implies $\mathsf{Baire\text{-}DRT}^k_\ell$. □

6.2. **The strength of "Every Borel set has the property of Baire".** We have just seen that over ATR_0, the Borel and Baire versions of the Dual Ramsey Theorem are equivalent. But only one direction used ATR_0, in order to assert that every Borel set has the property of Baire. In this section, we ask if this principle really requires ATR_0. We find that it does, but the reason is unsatisfactory, because it depends on a technicality in the standard definition of a Borel set. Some of the authors of the present paper removed that technicality in the later-researched but earlier-appearing paper [2]. When the technicality is removed, a principle strictly weaker than ATR_0 emerges. We refer the reader to [2] for details.

In this section we show:

Theorem 6.9 (RCA_0)**.** *The following are equivalent.*

1. ATR_0.
2. *For every Borel code B for a subset of $(\omega)^k$, there is an $x \in (\omega)^k$ such that $x \in B$ or $x \notin B$.*
3. *Every Borel code B for a subset of $(\omega)^k$ has a Baire code.*

In fact, the implication from (2) to (1) can be witnessed using only *trivial* Borel codes, which we define as follows.

Definition 6.10 (RCA_0)**.** A Borel code (B, φ) for a subset of $(\omega)^k$ is *trivial* if every leaf is labeled with either $\emptyset$ or $(\omega)^k$.

If B is a trivial Borel code, then an evaluation map for B at p is independent of p, so we can refer to an evaluation map f for B. Because we work with trivial Borel codes, the underlying topological space does not matter as long as Borel codes are defined in a manner similar to Definitions 6.1 and 6.2. For example, Theorem 6.9 holds for Borel codes of subsets of 2^ω or ω^ω as defined in Simpson [17]. (The fact that the leaves are labeled in Definition 6.1 does not affect any of the arguments in this section.)

The main ideas in the proof that (2) implies (1) use effective transfinite recursion and are similar to those in Section 7.7 of Ash and Knight [1].

Proposition 6.11 (RCA_0)**.** *The statement "every trivial Borel code has an evaluation map" implies* ACA_0*.*

Proof. Fix $g : \omega \to \omega$ and we show range(g) exists. Let B be the trivial Borel code consisting of the initial segments of $\langle n, m, 1\rangle$ for $g(m) = n$ and $\langle n, m, 0\rangle$ for $g(m) \neq n$. Label all leaves which end in 0 with $\emptyset$, and label all leaves which end in 1 with the entire space. Label all interior nodes with $\cup$. Let f be an evaluation map for B. Then $f(\langle n\rangle) = 1$ if and only if there is an m such that $g(m) = n$. □

In order to strengthen this result to imply ATR_0, we need to verify that effective transfinite recursion works in ACA_0. Let $LO(X)$ and $WO(X)$ be the standard formulas in second order arithmetic saying X is a linear order and X is a well order. We abuse notation and write $x \in X$ in place of $x \in \text{field}(X)$. For a formula $\varphi(n, X)$, $H_\varphi(X, Y)$ is the formula stating $LO(X)$ and $Y = \{\langle n, j\rangle : j \in X \wedge \varphi(n, Y^j)\}$ where $Y^j = \{\langle m, a\rangle : a <_X j \wedge \langle m, a\rangle \in Y\}$. When φ is arithmetic, $H_\varphi(X, Y)$ is arithmetic and ACA_0 proves that if $WO(X)$, then there is at most one Y such that $H_\varphi(X, Y)$. We define our formal version of effective transfinite recursion.

Definition 6.12. ETR is the axiom scheme

$$\forall X \left[\big(WO(X) \wedge \forall Y \, \forall n \, (\varphi(n,Y) \leftrightarrow \neg\psi(n,Y)) \big) \rightarrow \exists Y \, H_\varphi(X,Y) \right]$$

where φ and ψ range over Σ^0_1 formulas.

We show that ETR is provable in ACA_0. Following Simpson [17], we avoid using the recursion theorem and note that the only place the proof goes beyond RCA_0 is in the use of transfinite induction for Π^0_2 formulas, which holds is ACA_0 and is equivalent to transfinite induction for Σ^0_1 formulas. Greenberg and Montalbán [8] point out that ETR can also be proved using the recursion theorem, although this proof also uses Σ^0_1 transfinite induction.

Proposition 6.13. ETR *is provable in* ACA_0.

Proof. Fix a well order X and Σ^0_1 formulas φ and ψ. Throughout this proof, we let f, g and h be variables denoting finite partial functions from ω to $\{0,1\}$ coded in the canonical way as finite sets of ordered pairs. We write $f \preceq g$ (or $f \prec X$) if $f \subseteq g$ (or $f \subseteq \chi_X$) as sets of ordered pairs. By the usual normal form results (e.g. Theorem II.2.7 in Simpson), we fix a Σ^0_0 formula φ_0 such that

$$\forall Y \, \forall n \, \big(\varphi(n,Y) \leftrightarrow \exists f \, (f \prec Y \wedge \varphi_0(n,f))\big)$$

and such that if $\varphi_0(n,f)$ and $f \prec g$, then $\varphi_0(n,g)$. We fix a formula ψ_0 related to ψ in the same manner. Since $\varphi(n,Y) \leftrightarrow \neg\psi(n,Y)$, we cannot have compatible f and g such that $\varphi_0(n,f)$ and $\psi_0(n,g)$.

Our goal is to use partial functions f as approximations to a set Y such that $H_\varphi(X,Y)$. Therefore, we view $\mathrm{dom}(f)$ as consisting of coded pairs $\langle n,a\rangle$. For f to be a suitable approximation to Y, we need that if $\langle n,a\rangle \in \mathrm{dom}(f)$ and $a \notin X$, then $f(\langle n,a\rangle) = 0$. Similarly, if f is an approximation to Y^j, we need that $f(\langle n,a\rangle) = 0$ whenever $\langle n,a\rangle \in \mathrm{dom}(f)$ and $a \geq_X j$. These observations motivate the following definitions.

Let f be a finite partial function and let $i \in X$. We define

$$f^i = f \restriction \{\langle n,a\rangle : n \in \omega \wedge a <_X i\}.$$

We say $g \succeq f$ is an *i-extension of* f if for all $\langle n,a\rangle \in \mathrm{dom}(g) - \mathrm{dom}(f)$, $g(\langle n,a\rangle) = 0$ and either $a \notin X$ or $i \leq_X a$.

For $j \in X$, f is a *j-approximation* if the following conditions hold.

- If $\langle n,a\rangle \in \mathrm{dom}(f)$ with $a \notin X$ or $j \leq_X a$, then $f(\langle n,a\rangle) = 0$.
- If $\langle n,a\rangle \in \mathrm{dom}(f)$ and $a <_X j$, then
  - if $f(\langle n,a\rangle) = 1$, then there is an a-extension h of f^a such that $\varphi_0(n,h)$, and
  - if $f(\langle n,a\rangle) = 0$, then there is an a-extension h of f^a such that $\psi_0(n,h)$.

Note that if f is a j-approximation and $i <_X j$, then f^i is an i-approximation. Also, if f is a j-approximation and g is a j-extension of f, then g is a j-approximation.

Claim. For all $j \in X$, there do not exist $m \in \omega$ and j-approximations f and g such that $\varphi_0(m,f)$ and $\psi_0(m,g)$.

The proof is by transfinite induction on j. Fix the least $j \in X$ for which this property fails and fix witnesses m, f and g. To derive a contradiction, it suffices to show that f and g are compatible. Fix $\langle k,a\rangle$ such that both $f(\langle k,a\rangle)$ and $g(\langle k,a\rangle)$ are defined. If $a \notin X$ or $j \leq_X a$, then $f(\langle k,a\rangle) = g(\langle k,a\rangle) = 0$.

Suppose for a contradiction that $a <_X j$ and $f(\langle k,a\rangle) \neq g(\langle k,a\rangle)$. Without loss of generality, $f(\langle k,a\rangle) = 1$ and $g(\langle k,a\rangle) = 0$. Fix a-extensions h and h' of f^a and g^a respectively such that $\varphi_0(k,h)$ and $\psi_0(k,h')$. Since f is a j-approximation, f^a is an a-approximation, and since h is an a-extension of f^a, h is also an a-approximation. Similarly, h' is an a-approximation. Therefore, we have $k \in \omega$, $a <_X j$ and a-approximation h and h' such that $\varphi_0(k,h)$ and $\psi_0(k,h')$ contradicting the minimality of j.

Claim. For any j-approximation f and any $m \in \omega$, there is a j-approximation $g \succeq f$ such that either $\varphi_0(m,g)$ or $\psi_0(m,g)$.

The proof is again by transfinite induction on j. Fix the least j for which this property fails and fix witnesses f and m. Let $\langle n_s, i_s\rangle$ enumerate the pairs not in the domain of f. Below, we define a sequence $f = f_0 \preceq f_1 \preceq \cdots$ of j-approximations such that $f_{s+1}(\langle n_s, i_s\rangle)$ is defined. Let Y be the set with $\chi_Y = \cup_s f_s$. Either $\varphi(m,Y)$ or $\psi(m,Y)$ holds, and so there is a $g \prec Y$ such that $\varphi_0(m,g)$ or $\psi_0(m,g)$ holds. Fixing s such that $g \preceq f_s$ shows that either $\varphi_0(m,f_s)$ or $\psi_0(m,f_s)$ holds for the desired contradiction.

To define f_{s+1}, we need to extend f_s to a j-approximation f_{s+1} with $\langle n_s, i_s\rangle \in \mathrm{dom}(f_{s+1})$. We break into several cases. If $f_s(\langle n_s, i_s\rangle)$ is already defined, let $f_{s+1} = f_s$. Otherwise, if $i_s \notin X$ or $j \leq_X i_s$, set $f_{s+1}(\langle n_s, i_s\rangle) = 0$ and leave the remaining values as in f_s. In both cases, it is clear that f_{s+1} is a j-approximation.

Finally, if $i_s <_X j$ and $f_s(\langle n_s, i_s\rangle)$ is undefined, we apply the induction hypothesis to the i_s-approximation $f_s^{i_s}$ to get an i_s-approximation $g \succeq f_s^{i_s}$ such that either $\varphi_0(n_s, g)$ holds or $\psi_0(n_s, g)$ holds. Define f_{s+1} as follows.

- For $\langle m,a\rangle \in \mathrm{dom}(g)$ with $a <_X i_s$, set $f_{s+1}(\langle m,a\rangle) = g(\langle m,a\rangle)$.
- For $\langle m,a\rangle \in \mathrm{dom}(f_s)$ with $i_s \leq_X a$ or $a \notin X$, set $f_{s+1}(\langle m,a\rangle) = f_s(\langle m,a\rangle)$.
- Set $f_{s+1}(\langle n_s, i_s\rangle) = 1$ if $\varphi_0(n_s, g)$ holds and $f_{s+1}(\langle n_s, i_s\rangle) = 0$ if $\psi_0(n_s, g)$ holds.

It is straightforward to verify that $f_s \prec f_{s+1}$, g is an i_s-extension of $f_{s+1}^{i_s}$ and f_{s+1} is a j-approximation, completing the proof of the claim.

We define the set Y for which we will show $H_\varphi(X,Y)$ holds by $\langle m,j\rangle \in Y$ if and only if $j \in X$ and there is a j-approximation f such that $\varphi_0(m,f)$. It follows from the claims above that $\langle m,j\rangle \notin Y$ if and only if either $j \notin X$ or there is a j-approximation f such that $\psi_0(m,f)$. Therefore, Y has a Δ^0_1 definition. The next two claims show that $H_\varphi(X,Y)$ holds, completing our proof.

Claim. If f is a j-approximation, then $f \prec Y^j$.

Consider $\langle m,a\rangle \in \mathrm{dom}(f)$. If $a \notin X$ or $j \leq_X a$, then $f(\langle m,a\rangle) = Y^j(\langle m,a\rangle) = 0$. Suppose $a <_X j$. If $f(\langle m,a\rangle) = 1$, then there is an a-extension g of f^a such that $\varphi_0(m,g)$. Since f^a is an a-approximation and g is an a-extension of f^a, g is an a-approximation. Therefore, $\langle m,a\rangle \in Y$ by definition and hence $\langle m,a\rangle \in Y^j$. By similar reasoning, if $f(\langle m,a\rangle) = 0$, then $\langle m,a\rangle \notin Y$ and hence $\langle m,a\rangle \notin Y^j$.

Claim. $\langle m,j\rangle \in Y$ if and only if $\varphi(m,Y^j)$.

Assume that $\langle m,j\rangle \in Y$ and fix a j-approximation f such that $\varphi_0(m,f)$. Since $f \prec Y^j$, $\varphi(m,Y^j)$. For the other direction, assume that $\varphi(m,Y^j)$. Fix a j-approximation f such that either $\varphi_0(m,f)$ or $\psi_0(m,f)$. Since $f \prec Y^j$ and $\varphi(m,Y^j)$, we must have $\varphi_0(m,f)$ and therefore $\langle m,j\rangle \in Y$ by definition. □

We recall some notation and facts from Simpson [17] to state the equivalence of ATR_0 we will prove. We let $TJ(X)$ denote the Turing jump in ACA_0 given by fixing a universal Π^0_1 formula. We use the standard recursion theoretic notations Φ^X_e and $\Phi^X_{e,s}$ with the understanding that they are defined by this fixed universal formula.

$\mathcal{O}_+(a, X)$ is the arithmetic statement that $a = \langle e, i\rangle$, e is an X-recursive index of an X-recursive linear order $\leq^X_e$ and $i \in \text{field}(\leq^X_e)$. $\mathcal{O}^X_+ = \{a : \mathcal{O}_+(a, X)\}$ exists in ACA_0. For $a, b \in \mathcal{O}^X_+$, we write $b <^X_{\mathcal{O}} a$ if $a = \langle e, i\rangle$, $b = \langle e, j\rangle$ and $j <^X_e i$. For $a \in \mathcal{O}^X_+$, the set $\{b : b <^X_{\mathcal{O}} a\}$ exists in ACA_0.

$\mathcal{O}(a, X)$ is the Π^1_1 statement $\mathcal{O}_+(a, X) \wedge WO(\{b : b <^X_{\mathcal{O}} a\})$. Intuitively, $\mathcal{O}(a, X)$ says that $a = \langle e, i\rangle$ is an X-recursive ordinal notation for the well ordering given by the restriction of $\leq^X_e$ to $\{j : j <^X_e i\}$. In ATR_0, if $\mathcal{O}(a, X)$, then the set

$$H^X_a = \{\langle y, 0\rangle : y \in X\} \cup \{\langle y, b + 1\rangle : b <^X_{\mathcal{O}} a \wedge y \in TJ(H^X_b)\}$$

exists. In ACA_0, there is an arithmetic formula $H(a, X, Y)$ which, under the assumption that $\mathcal{O}(a, X)$, holds if and only if $Y = H^X_a$.

By Theorem VIII.3.15 in Simpson [17], ATR_0 is equivalent over ACA_0 to

$$\forall X\, \forall a\, (\mathcal{O}(a, X) \rightarrow H^X_a \text{ exists}).$$

If $\mathcal{O}(a, X)$ with $a = \langle e, i\rangle$, then we can assume without loss of generality that there are a' and a'' such that $\mathcal{O}(a', X)$, $\mathcal{O}(a'', X)$ and $a <^X_{\mathcal{O}} a' <^X_{\mathcal{O}} a''$ by adding two new successors of i in $\leq^X_e$ if necessary. Therefore, to prove ATR_0, it suffices to fix a and X such that $\mathcal{O}(a, X)$ and prove $\forall c <^X_{\mathcal{O}} b\, (H^X_c \text{ exists})$ for each $b <^X_{\mathcal{O}} a$.

Theorem 6.14 (ACA_0)**.** *The statement "every trivial Borel code has an evaluation map" implies* ATR_0*.*

Proof. Fix a and X such that $\mathcal{O}(a, X)$, so the restriction of $<^X_{\mathcal{O}}$ to $\{b : b <^X_{\mathcal{O}} a\}$ is a well order. Using ETR, we define trivial Borel codes $B_{x,b}$ for $x \in \omega$ by transfinite recursion on $b <^X_{\mathcal{O}} a$. We explain the intuitive construction before the formal definition.

Let $b <^X_{\mathcal{O}} a$ and $x \in \omega$. We want to define a trivial Borel code $B_{x,b}$ such that if f is an evaluation map for $B_{x,b}$, then $f(\lambda) = 1$ if and only if $x \in TJ(H^X_b)$. We label λ with $\cup$. For each binary string σ such that $\Phi^\sigma_{x,|\sigma|}(x)$ converges, we add a successor $\langle n_\sigma\rangle$. Here $\sigma \mapsto n_\sigma$ is just some primitive recursive bijection between $2^{<\omega}$ and ω. It follows that $f(\lambda) = 1$ if and only if there is a σ such that $\Phi^\sigma_{x,|\sigma|}(x)$ converges and $f(\langle n_\sigma\rangle) = 1$. (In case $\Phi^\sigma_{x,|\sigma|}(x)$ always diverges, we may also add a leaf $\langle n\rangle$ which is labeled with $\emptyset$. In this case, $f(\lambda) = f(\langle n\rangle) = 0$ and $x \notin TJ(H^X_b)$ which is what we want.)

Next, we want to ensure $f(\langle n_\sigma\rangle) = 1$ if and only if $\sigma \prec H^X_b$. We label $\langle n_\sigma\rangle$ with $\cap$, and for each $k < |\sigma|$, we add a successor $\langle n_\sigma, k\rangle$. We want $f(\langle n_\sigma, k\rangle) = 1$ if and only if $\sigma(k) = H^X_b(k)$. We break into cases to determine the extensions of $\langle n_\sigma, k\rangle$.

For the first case, suppose $k = \langle y, 0\rangle$. We want $f(\langle n_\sigma, k\rangle) = 1$ if and only if $y \in X$. If $\sigma(k) = X(y)$, we label this node with the entire space, and if $\sigma(k) \neq X(y)$, we label this node with $\emptyset$. In either case, the successor nodes will be leaves so we have $f(\langle n_\sigma, k\rangle) = 1$ if and only if $k \in H^X_b$.

For the second case, suppose $k = \langle y, c + 1\rangle$ and $c <^X_{\mathcal{O}} b$. By the induction hypothesis, we have defined the trivial Borel code $B_{y,c}$ already. If $\sigma(k) = 1$, then we label $\langle n_\sigma, k\rangle$ with $\cup$, and attach to it a copy of $B_{y,c}$, treating $\langle n_\sigma, k\rangle$ as the root of $B_{y,c}$. The map f restricted to the subtree above $\langle n_\sigma, k\rangle$ is an evaluation map for

$B_{y,c}$ and hence by the inductive hypothesis

$$f(\langle n_\sigma, k\rangle) = 1 \Leftrightarrow y \in TJ(H_c^X) \Leftrightarrow k \in H_b^X \Leftrightarrow \sigma(k) = H_b^X(k).$$

On the other hand, if $\sigma(k) = 0$, then we label $\langle n_\sigma, k\rangle$ with $\cap$ and extend it by a copy of $\overline{B}_{y,c}$. By the inductive hypothesis, we have

$$f(\langle n_\sigma, k\rangle) = 1 \Leftrightarrow y \notin TJ(H_c^X) \Leftrightarrow k \notin H_b^X \Leftrightarrow \sigma(k) = H_b^X(k).$$

For the third case, suppose that $k = \langle y, c+1\rangle$ and $c \not<_{\mathcal{O}}^X b$. In this case, we know $H_b^X(k) = 0$. If $\sigma(k) = 0$, we label $\langle n_\sigma, k\rangle$ with the entire space, and if $\sigma(k) = 1$ we label it with $\emptyset$.

The formal construction follows this outline. To simplify the notation, for a trivial Borel code B, we let $B^1 = B$ and $B^0 = \overline{B}$. Since "$\Phi^\sigma_{x,|\sigma|}(x)$ converges" is a bounded quantifier statement and $c <_{\mathcal{O}}^X b$ is a Δ^0_1 statement with parameter X, the following recursion on $b <_{\mathcal{O}}^X a$ can be done with ETR. For each $x \in \omega$, we put λ in $B_{x,b}$ with label $\cup$. For each σ such that $\Phi^\sigma_{x,|\sigma|}(x)$ converges, we put $\langle n_\sigma\rangle$ and $\langle n_\sigma, k\rangle$ in $B_{x,b}$ for all $k < |\sigma|$. We label $\langle n_\sigma\rangle$ with $\cap$. We extend $\langle n_\sigma, k\rangle$ as follows.

- For $k = \langle y, 0\rangle$: if $\sigma(k) = X(y)$, then $\langle n_\sigma, k\rangle$ is labeled with the whole space, and if $\sigma(k) \neq X(y)$, then it is labeled with $\emptyset$.
- For $k = \langle y, c+1\rangle$ with $c <_{\mathcal{O}}^X b$, $\langle n_\sigma, k\rangle^\frown\tau \in B_{x,b}$ for all $\tau \in B_{y,c}^{\sigma(k)}$, with labels inherited from $B_{y,c}^{\sigma(k)}$.
- For $k = \langle y, c+1\rangle$ with $c \not<_{\mathcal{O}}^X b$, $\langle n_\sigma, k\rangle$ gets labeled with the whole set if $\sigma(k) = 0$ and labeled with $\emptyset$ if $\sigma(k) = 1$.

This completes the construction of the trivial Borel codes $B_{x,b}$ for $b <_{\mathcal{O}}^X a$ by ETR. To complete the proof, we fix an arbitrary $b <_{\mathcal{O}}^X a$ and show that $\forall c <_{\mathcal{O}}^X b\,(H_c^X \text{ exists})$.

Fix an index x and $s \in \omega$ such that $\Phi^{1^s}_{x,s}(x)$ converges. Let N be the least value of s witnessing this convergence so $\Phi^{1^s}_{x,s}(x)$ converges for all $s \geq N$. Let f be an evaluation map for $B_{x,b}$.

For $c <_{\mathcal{O}}^X b$ and $y \in \omega$, let $\sigma = 1^{N+k}$ where $k = \langle y, c+1\rangle$. Define $f_{y,c}(\tau) = f(\langle n_\sigma, k\rangle^\frown\tau)$. We claim $f_{y,c}$ is an evaluation map for $B_{y,c}$. By the choice of x, $\Phi^\sigma_{x,|\sigma|}(x)$ converges. Since $c <_{\mathcal{O}}^X b$ and $\sigma(k) = 1$, we have $\langle n_\sigma, k\rangle^\frown\tau \in B_{x,b}$ if and only if $\tau \in B_{y,c}$. Therefore, $f_{y,c}$ is defined on $B_{y,c}$ and it satisfies the conditions for an evaluation map because f does.

Recall that $H(x, X, Y)$ is a fixed arithmetic formula such that if $\mathcal{O}(x, X)$, then $H(x, X, Y)$ holds if and only if $Y = H_x^X$. Define

$$Z = \{\langle y, 0\rangle : y \in X\} \cup \{k : k = \langle y, c+1\rangle \wedge c <_{\mathcal{O}}^X b \wedge f(\langle n_\sigma, k\rangle) = 1\}.$$

For $c <_{\mathcal{O}}^X b$, let $Z^c = \{\langle y, r\rangle \in Z : r = 0 \vee r - 1 <_{\mathcal{O}}^X c\}$. We show the following properties by simultaneous arithmetic induction on $c <_{\mathcal{O}}^X b$.

(1) $H(c, X, Z^c)$ holds. That is, $Z^c = H_c^X$.
(2) For all y, $f_{y,c}(\lambda) = 1$ if and only if $y \in TJ(Z^c) = TJ(H_c^X)$.

These properties imply $\forall c <_{\mathcal{O}}^X b\,(H_c^X \text{ exists})$ completing our proof.

Fix $c <_{\mathcal{O}}^X b$ and assume (1) and (2) hold for $d <_{\mathcal{O}}^X c$. To see (1) holds for c, fix k. If $k = \langle y, 0\rangle$, then $k \in Z^c \Leftrightarrow y \in X \Leftrightarrow k \in H_c^X$. Suppose $k = \langle y, d+1\rangle$. If $d \not<_{\mathcal{O}}^X c$, then $k \notin H_c^X$ and $k \notin Z^c$. If $d <_{\mathcal{O}}^X c$, then

$$k \in Z^c \Leftrightarrow f(\langle n_\sigma, k\rangle) = 1 \Leftrightarrow f_{y,d}(\lambda) = 1.$$

By the induction hypothesis, $k \in Z^c$ if and only if $y \in TJ(Z^d) = TJ(H_d^X)$, which holds if and only if $k \in H_c^X$, completing the proof of (1).

To prove (2), fix y and let $k = \langle y, c+1 \rangle$. By definition,

$$k \in Z^c \Leftrightarrow f_{y,c}(\lambda) = f(\langle n_\sigma, k \rangle) = 1,$$

and $y \in TJ(Z^c) = TJ(H_c^X)$ if and only if there is a σ such that $\Phi^\sigma_{y,|\sigma|}(y)$ converges and $\sigma \prec Z^c = H_c^X$.

Suppose there are no σ such that $\Phi^\sigma_{y,|\sigma|}(y)$ converges. In this case, $y \notin TJ(H_c^X)$ and $f_{y,c}(\lambda) = 0$. Therefore $f_{y,c}(\lambda) = 1$ if and only if $y \in TJ(H_c^X)$ as required.

Suppose $\Phi^\sigma_{y,|\sigma|}(y)$ converges for some σ. For any such σ, $\langle n_\sigma, k \rangle \in B_{y,c}$ for all $k < |\sigma|$. By the induction hypothesis and the case analysis in the intuitive explanation of the construction, we have $f_{y,c}(\langle n_\sigma \rangle) = 1$ if and only if $\sigma \prec H_c^X = Z^c$, and therefore, $f_{y,c}(\lambda) = 1$ if and only if there is a σ such that $\Phi^\sigma_{y,|\sigma|}(y)$ converges and $\sigma \prec H_c^X$, completing the proof of (2) and of the theorem. □

We conclude with a proof of Theorem 6.9.

Proof. Lemma V.3.3 in Simpson [17] shows (1) implies (2) in the space 2^ω and the proof translates immediately to $(\omega)^k$. By Proposition 6.5, (1) implies (3). It follows from Theorem 6.14 that (2) implies (1). We show (3) implies (2). Let B be a Borel code. Fix a Baire code U, V and D_n for B. Since each D_n and $U \cup V$ is a dense open set, there is an $x \in (U \cup V) \cap \bigcap_{n \in \omega} D_n$. If $x \in U$, then by the definition of a Baire code, $x \in B$, and similarly, if $x \in V$, then $x \notin B$. Therefore, we have a partition x such that $x \in B$ or $x \notin B$ as required. □

7. Open Questions

While Figure 1.1 summarizes the known implications among the studied principles, in most cases it is not known whether the results are optimal. It is particularly dissatisfying that the best upper bound for these principles remains Π^1_1-CA_0. Observe that, on the basis of the proof of CDRT^k_ℓ given in Theorem 3.18, any upper bound on the strength of the Carlson-Simpson Lemma $\mathsf{CSL}(k-1, \ell)$ would also imply a related upper bound on the strength of CDRT^k_ℓ. Therefore, it would be interesting to know the following:

Question 7.1. For any $k \geq 3$, does $\mathsf{CSL}(k, \ell)$ follow from ATR_0?

The best known upper bound for $\mathsf{CSL}(2, \ell)$ is ACA_0; it is shown in [11] that the stronger principle $\mathsf{OVW}(2, \ell)$ follows from ACA_0.

Turning attention now to lower bounds, the principles CDRT^k_ℓ for $k \geq 4$ are not obviously implied by HT or ACA_0^+. We wonder whether an implication might go the other way.

Question 7.2. For any $k \geq 4$, does CDRT^k_ℓ imply HT or ACA_0^+?

When $k \geq 4$, it is known that CDRT^k_ℓ implies ACA_0 (this was proved for ODRT^k_ℓ in [12]). On the other hand, while CDRT^3_2 is provable from Hindman's Theorem, the best lower bound we have on CDRT^3_2 is RT^2_2. Furthermore, nothing about the relationship of CDRT^3_2 and ACA_0 is known.

Question 7.3. Is CDRT^3_2 comparable to ACA_0?

For the $k = 2$ case, can Theorem 5.7 be strengthened in the following way?

Question 7.4. Is Δ^0_n-$\mathsf{DRT}^2_2 \equiv_{\mathrm{sW}} D^n_2$?

These are just a few of the many questions that remain concerning these principles.

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