COMPLETELY DETERMINED BOREL SETS AND MEASURABILITY

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Abstract. We consider the reverse math strength of the statement $\mathsf{CD\text{-}M}$: "Every completely determined Borel set is measurable." Over WWKL, we obtain the following results analogous to the previously studied category case:
(1) $\mathsf{CD\text{-}M}$ lies strictly between ATR_0 and $\mathsf{L}_{\omega_1,\omega}\text{-}\mathsf{CA}$.
(2) Whenever $M \subseteq 2^\omega$ is the second-order part of an ω-model of $\mathsf{CD\text{-}M}$, then for every $Z \in M$, there is a $R \in M$ such that R is Δ^1_1-random relative to Z.

On the other hand, without WWKL, all sets have measure zero and thus $\mathsf{CD\text{-}M}$ loses its meaning. Vacuously, $\neg\mathsf{WWKL}$ implies $\mathsf{CD\text{-}M}$ over RCA_0.

1. Introduction

The notion of a *completely determined Borel set* was introduced in [ADM+19] to permit the reverse mathematics analysis of weak principles involving Borel sets. In the standard treatment of Borel sets in reverse mathematics [Sim09], a Borel set is any well-founded tree T whose leaves are labeled with clopen sets and whose interior nodes are labeled with intersections or unions. A real $X \in 2^\omega$ is then said to belong to the set coded by T if and only if there is an *evaluation map*, a function $f : T \to \{0,1\}$ such that $f(\sigma) = 1$ if and only if X is in the set coded by $T_\sigma := \{\tau : \sigma^\frown\tau \in T\}$. While *arithmetic transfinite recursion* (ATR_0) suffices to construct evaluation maps for each X, in general it is also required. As a result, most principles that even mention a Borel set reverse to ATR_0 simply because most such principles have a conclusion that presupposes an element X in the Borel set.

An exception was encountered by [DFSW17] in their analysis of the Borel dual Ramsey theorem. The hypothesis of this theorem posits ℓ-many Borel sets whose union is the entire space. In order to say the union is the entire space, the existence of evaluation maps for each X must be a part of the hypothesis. That is, an instance of the Borel dual Ramsey theorem is not well-defined unless the given Borel sets are *completely determined*, meaning that each X has an evaluation map.

This example fueled the idea that the lack of interesting reversals for weak principles involving Borel sets could be remedied by restricting attention to completely determined Borel sets. This was borne out in [ADM+19], in which the following was proven about the principle $\mathsf{CD\text{-}PB}$: "Every completely determined Borel set has the property of Baire."

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Theorem 1.1 ([ADM$^+$19]). *The principle* $\mathsf{CD\text{-}PB}$ *is strictly weaker than* ATR_0. *Every* ω*-model* $\mathcal{M}$ *of* $\mathsf{CD\text{-}PB}$ *is closed under hyperarithmetic reduction, and for every* $Z \in M$*, there is some* $G \in \mathcal{M}$ *that is* $\Delta^1_1(Z)$*-generic.*

In this paper we do the same for the principle "every Borel set is measurable." Similar results are obtained by similar methods. The only new twist is the need to work with an appropriate meaning of "measurable" for a Borel set; there are several candidates. This delicate task has already been undertaken by Simpson, X. Yu, Brown, Giusto and others (see for example [Sim09, Chapter X], [Yu93], [Yu94], and [BGS02]). We summarize their work and give the sometimes more detailed versions of the results needed for our application.

We then define the principle $\mathsf{CD\text{-}M}$: "Every completely determined Borel set is measurable." We show that $\mathsf{CD\text{-}M}$ follows from $\neg\mathsf{WWKL}$ (for the simple reason that $\neg\mathsf{WWKL}$ implies the Cantor space has measure 0, and thus every subset of it is also measure 0). On the other hand, working over WWKL, we obtain results similar to the category case.

In [ADM$^+$19], a model was constructed in which a Baire approximation to a given completely determined Borel set B was obtained without ATR_0 by polling $\Sigma^1_1(B)$-generics about their membership in B. We do essentially the same to construct a proof of measurability of a given completely determined set B, but using $\Pi^1_1(B)$-randoms. The result of this polling is exactly an element $f \in L^1(2^\omega)$, so no translation is required to obtain a code for a measurable set as defined in [Sim09, Chapter X]. The main results of this paper are as follows.

Theorem 1.2. *The principle* $\mathsf{CD\text{-}M}$ *is strictly weaker than* ATR_0. *Every* ω*-model* $\mathcal{M}$ *of* $\mathsf{CD\text{-}M}$ *is closed under hyperarithmetic reduction, and for every* $Z \in \mathcal{M}$*, there is some* $R \in \mathcal{M}$ *that is* $\Delta^1_1(Z)$*-random.*

These results were first presented by the author at the Institute for Mathematical Sciences workshop Higher Recursion Theory and Set Theory, using a version of Proposition 4.5 to quickly move the base theory to ACA_0, and using an ad hoc notion of a "function measuring a set" which was later found to essentially coincide with the notion of a measurable characteristic function previously proposed by Simpson and several of his students. The author would like to thank Steve Simpson for his suggestion to lower the base theory and for bringing that connection to light. Finally, the author would like to thank Ted Slaman, her PhD advisor, for his support and mentorship, his good humor and sound principles, and his excellent body of research which this volume celebrates.

2. Notation and Preliminaries

We use the notation and conventions of [ADM$^+$19]. In that paper, much more background and context can be found in the introduction. The eth Turing functional is denoted Φ_e. Elements of $\omega^{<\omega}$ are denoted by σ, τ and elements of $2^{<\omega}$ by p, q. We write $\sigma \preceq \tau$ to indicate that σ is an initial segment of τ, with $\prec$ if $\sigma \neq \tau$. For $p \in 2^{<\omega}$, the notation $[p]$ refers to the cylinder $\{X \in 2^\omega : p \prec X\}$. The empty string is denoted by λ. A string with a single component of value $n \in \omega$ is denoted by $\langle n \rangle$. String concatenation is denoted by $\sigma\tau$. Usually we write σn instead of the more technically correct but uglier $\sigma\langle n \rangle$.

If U is a set of strings (for example, a tree, or a coded open subset of 2^ω), and σ is any string, we write $\sigma^\frown U$ to mean $\{\sigma\tau : \tau \in U\}$. If T is a tree and $\sigma \in T$, we write T_σ to mean $\{\tau : \sigma\tau \in T\}$, and if $\langle n\rangle \in T$, we write T_n to mean $\{\tau : n\tau \in T\}$.

We assume familiarity with reverse mathematics, in particular the systems RCA_0, WWKL, ACA_0 and ATR_0. We note that effective transfinite recursion and arithmetic transfinite induction can be carried out in ACA_0. We identify an ω-model $\mathcal{M}$ of second order arithmetic with its second-order part, writing $X \in \mathcal{M}$ to mean that X is an element of the second-order part of $\mathcal{M}$.

We assume familiarity with ordinal notations and pseudo-ordinals. Kleene's O is denoted by $\mathcal{O}$. The relation $<_*$ is the transitive closure of the relation defined by $1 <_* x$ if $x \neq 1$, $x <_* 2^x$, and $\Phi_e(n) <_* 3 \cdot 5^e$. We will not distinguish between ordinals and their notations. Additionally, if $b \in \mathcal{O}$, we write $b+1$ for the successor of b (rather than the more technically correct but cumbersome 2^b) and $b+O(1)$ for the outcome of taking some fixed constant number of successors of b. If $b \in \mathcal{O}$ the unique jump hierarchy on b is denoted H_b. All these concepts can be relativized to an oracle Z. Kleene's $\mathcal{O}$ also has a Σ^1_1 superset $\mathcal{O}^*$, defined as the intersection of all $X \in HYP$ such that $1 \in X$, $a \in X \implies 2^a \in X$, and

$$\forall n[\Phi_e(n) \in X \text{ and } \Phi_e(n) <_* \Phi_e(n+1)] \implies 3 \cdot 5^e \in X.$$

Observe also that $\mathcal{O}$ is contained in $\mathcal{O}^*$. The elements of $\mathcal{O}^* \setminus \mathcal{O}$ are called pseudo-ordinals. For more details, see the introduction of [ADM+19].

A $T \subseteq \omega^{<\omega}$ is well-founded if it has no infinite path. If T is any tree, and $\rho : T \to \mathcal{O}^*$, we say that ρ *ranks* T if for all σ and n such that $\sigma^\frown n \in T$, we have $\rho(\sigma^\frown n) <_* \rho(\sigma)$, and for each leaf $\sigma \in T$, $\rho(\sigma) = 1$. If T is ranked by ρ and $\rho(\lambda) = a$, we say that T is *a-ranked* by ρ. If $a \in \mathcal{O}$ and T is a-ranked then T is well-founded, but it is possible and useful for an ill-founded tree to be ranked by a pseudo-ordinal. A tree T is *alternating* if whenever $\sigma \in T$ is a $\cap$, then each $\sigma n \in T$ is either a $\cup$ or a leaf, and similarly if $\sigma \in T$ is a $\cup$, then each $\sigma n \in T$ is either a $\cap$ or a leaf.

A labeled *Borel code* is a well-founded tree $T \subseteq \omega^{<\omega}$ whose leaves are labeled by basic open sets or their complements, and whose inner nodes are labeled by $\cup$ or $\cap$. The Borel set associated to a Borel code is defined by induction, interpreting the labels in the obvious way. Any Borel set can be represented this way, by applying DeMorgan's laws to push complementation out to the leaves. A formula of $L_{\omega_1,\omega}$ is a well-founded tree whose interior nodes are labeled with $\cap$ (conjunction) and $\cup$ (disjunction) and whose leaves are labeled with the symbols `true` or `false`.

There is a computable procedure which, for any $b \in \mathcal{O}$ and any $n \in \omega$, outputs a $b+O(1)$-ranked alternating formula of $L_{\omega_1,\omega}$ which holds true if and only if $n \in H_b$.

If T is a labeled Borel code and $X \in 2^\omega$, an *evaluation map* for $X \in T$ is a function $f : T \to \{0,1\}$ such that

- If σ is a leaf, $f(\sigma) = 1$ if and only if X is in the clopen set coded by $\ell(\sigma)$.
- If σ is a union node, $f(\sigma) = 1$ if and only if $f(\sigma^\frown n) = 1$ for some $n \in \omega$.
- If σ is an intersection node, $f(\sigma) = 1$ if and only if $f(\sigma^\frown n) = 1$ for all $n \in \omega$.

We say that X is in the set coded by T, denoted $X \in |T|$, if there is an evaluation map f for X in T such that $f(\lambda) = 1$. Note that $X \in |T|$ is a Σ^1_1 statement. In ACA_0, evaluation maps are unique when they exist. If T is ill-founded, the notation $|T|$ may not have meaning outside of a given model. If T is a truly well-founded

Borel code, we do use $|T|$ outside of the context of a model to denote the elements of the set that T codes.

A Borel code T is *completely determined* if every $X \in 2^\omega$ has an evaluation map in T. A formula ϕ of $L_{\omega_1,\omega}$ is *completely determined* if there is map $f : \phi \to \{\texttt{true}, \texttt{false}\}$ that agrees with ϕ on the leaves and satisfies the logic of ϕ at interior nodes. The principle $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ states that whenever $\langle \phi_n \rangle_{n\in\omega}$ is a sequence of completely determined formulas of $L_{\omega_1,\omega}$, then $\{n : \phi_n \text{ is true}\}$ exists.

We assume familiarity with higher randomness. The key theorems we need are:

Theorem 2.1 ([Ste73, Ste75])**.** *A real $R \in 2^\omega$ is Π^1_1-random if and only if it is Δ^1_1-random and $\omega_1^R = \omega_1^{ck}$.*

Theorem 2.2 ([HN07])**.** *For $R_0, R_1 \in 2^\omega$, we have $R_0 \oplus R_1$ is Π^1_1-random if and only if R_0 and R_1 are relatively Π^1_1-random.*

Theorem 2.3 ([CNY08])**.** *If $R_0 \oplus R_1$ is Π^1_1-random, then $\Delta^1_1(R_0) \cap \Delta^1_1(R_1) = \Delta^1_1$.*

3. Measure theory in reverse mathematics

Historically, measure theory developed as a third-order theory. Classically, a measure is a set function from a σ-algebra of subsets of a space to the non-negative reals. Therefore, although much of measure theory can be developed within second-order arithmetic, this development has required some care and some non-trivial choices. We now summarize work of Simpson, X. Yu, Brown, and Giusto [Yu90, YS90, Yu93, Yu94, BGS02, Sim09], in which this development took place.

In the context of second-order arithmetic, all the relevant information about a measure space $(X, \mu, \mathcal{S})$ is already contained in the values that μ takes on an algebra which generates $\mathcal{S}$ as a σ-algebra. When X is a separable complete metric space space and $\mathcal{S}$ is the Borel sets, a countable generating algebra is naturally obtained by taking all finite Boolean combinations of basic open sets. In the case of Cantor space 2^ω, this approach works out very cleanly because the basic open sets (and thus all elements of the generating algebra) are clopen. However, for an arbitrary separable complete metric space, a problem arises. What if there is an atom on the boundary of a basic open set U? Is it fair to ask that our encoding of a measure μ be able to precisely compute $\mu(U)$ and $\mu(U^c)$? (Because a typical open set V can only be represented as an infinite enumeration of its basic open subsets, its measure $\mu(V)$ would be at best c.e., not computable, in a description of μ and V.) Another way of asking the same question is: for the purposes of constructive mathematics, what is a suitable topology to put on the space of Borel measures on X?

When X is Cantor space, a popular representation choice has been to to name a measure μ with a function from $2^{<\omega}$ to $\mathbb{R}$ which records the measure of each basic clopen set (see for example [DM13]). This representation induces the so-called weak topology on the space of probability measures on X (see for example [Bog07, Definition 8.2.1]). This is the same topology induced by the Prohorov metric (see for example [Bog07, Theorem 8.3.2]), and also coincides with the weak-$*$ topology on $C(X)^*$ (see the discussion following Definition 8.2.1 in [Bog07]). Restricting attention to probability measures on compact complete separable metric spaces, Yu also settled on the same topology in [Yu93], and made the following definition.

Definition 3.1. *Let X be a compact complete separable metric space. A* Borel probability measure *μ on X is a bounded positive linear functional $\mu : C(X) \to \mathbb{R}$ with $\mu(1) = 1$.*

Here $C(X)$ denotes the Banach space of continuous real-valued functions on X with the supremum norm, and $1 \in C(X)$ denotes the constant function. Care is required in the definition of $C(X)$. It is not simply the collection of continuous function on X equipped with the supremum norm, because in weak subsystems of second-order arithmetic, a continuous function on a compact space X need not have a supremum. Instead, $C(X)$ is defined as a complete separable metric space by choosing a particularly well-behaved collection of continuous functions to be the dense subset. The details are given in [Sim09, Exercise 4.2.13], in which it is also established that $C(X)$ consists of precisely those continuous functions from X to $\mathbb{R}$ which also possess a modulus of uniform continuity. Therefore, while a measure μ on X is defined by specifying how to integrate elements of $C(X)$ with respect to μ, it does not follow that every continuous function on X is μ-integrable; only those with a modulus of uniform continuity come with this guarantee.

An unavoidable drawback to Definition 3.1 is that it puts a small distance between the definition of a measure and its basic function of assigning sizes to sets. Therefore, it is necessary to make a further definition for "the measure of an open set" (and subsequently a further definition for the measure of an arithmetic set, etc. leading up to the notion of a measurable set). At each point of definition, a choice arises: should the measure assignment be *intensional* (depending only on the *description* of the set in question) or *extensional* (depending on only on the *membership* of the set in question)?

To understand the tension here, consider that if U is any component of a universal Martin-Löf test in Cantor space with its usual fair-coin measure, then statement $U = 2^\omega$ holds in REC. Thus in REC, we cannot simultaneously have both of these two desirable properties:

(1) If $S \subseteq 2^{<\omega}$ is prefix-free, then $\mu\left(\bigcup_{\sigma\in S}[\sigma]\right) = \sum_{\sigma\in S} 2^{-|\sigma|}$
(2) If $A = B$ then $\mu(A) = \mu(B)$.

Note that the first is an intensional property and the second is an extensional property. Although both are clearly wanted, the second seems more essential. Thus the extensional definition for the measure of an open set is the one which appears in [Sim09].

Definition 3.2 (RCA_0)**.** *Let μ be a Borel probability measure on X. Let U be an open subset of X. The* μ-measure *of U is defined as*

$$\mu(U) = \sup\{\mu(f) : f \in C(X), 0 \leq f \leq 1, f(x) = 0 \text{ for } x \in X \setminus U\}.$$

In the absence of ACA_0, this supremum may not exist as a number, but statements about $\mu(U)$ may still be made in weaker systems by simply substituting the above definition of $\mu(U)$ in any sentence which makes a claim about this quantity. For example, it holds in RCA_0 that $U \subseteq V$ implies that $\mu(U) \leq \mu(V)$. Such statements are said to hold in a "virtual" or "comparative" sense.

Observe that this extensional definition also gives the "right" values on Cantor space with the fair coin measure when U is a finite union of non-intersecting cylinders $U = \cup_{i<n}[p_i]$. That is, $\mu(\cup_{i<n}[p_i]) = \sum_{i<n} 2^{-|p_i|}$.

On the other hand, in RCA_0 we can always assume that open subsets of Cantor space are given by prefix-free enumerations of elements of $2^{<\omega}$, so we can also give the following intensional definition of measure of an open set in Cantor space:

Definition 3.3 (RCA_0)**.** *If U is an open subset of 2^ω given by $U = \cup_{i<\omega}[p_i]$, where each $p_i \in 2^{<\omega}$ and where $\{p_i : i \in \omega\}$ is prefix-free, then define the* intensional measure *of U by $\mu_I(U) = \sum_i 2^{-|p_i|}$.*

The intensional and extensional definitions fully coincide under WWKL.

Theorem 3.4 ([YS90]; see also [BGS02])**.** *Over RCA_0, WWKL is equivalent to the statement that for every compact separable metric space X and every measure μ on X, μ is countably additive. That is, for every sequence of open sets U_n,*

$$\lim_N \mu(\cup_{n<N} U_n) = \mu(\cup_n U_n).$$

Corollary 3.5 (WWKL)**.** *For all open sets $U \subseteq 2^\omega$, $\mu(U) = \mu_I(U)$.*

One final intensional notion of a measurable set is needed for the development of measure theory.

Definition 3.6. *A* rapidly null G_δ *set is a G_δ set $\cap_n U_n$ such that for each n, $\mu_I(U_n) < 2^{-n}$.*

Note: a Martin-Löf test is just a computably presented rapidly null G_δ set.

Theorem 3.7 ([ADR12])**.** *Over RCA_0, WWKL is equivalent to the statement that if A is a rapidly null G_δ subset of 2^ω, then $A \neq 2^\omega$.*

Thus in WWKL, a μ-measurable set may be non-vacuously defined as follows. Let $\mu : C(X) \to \mathbb{R}$ be a positive Borel probability measure. Let $L^1(X,\mu)$ denote the completion of $C(X)$ with respect to the L^1 norm defined by $||f-g||_1 = \int |f-g|$. Recall that a sequence $\langle x_n \rangle$ of points of a metric space is called *rapidly Cauchy* if for all n, we have $d(x_n, x_{n+1}) < 2^{-n}$. Each element of $L^1(X,\mu)$ is represented by many *names*, where a name is a sequence $\langle f_n \rangle_{n\in\omega}$ of functions from $C(X)$ that is rapidly Cauchy for the L^1 norm.

Definition 3.8 ([BGS02])**.** *A* measurable characteristic function *is a function $f \in L^1(X,\mu)$ such that $f(x) \in \{0,1\}$ for all x outside a rapidly null G_δ set. A set E is* measurable *if there is some $f \in L^1(X,\mu)$ such that $f = \chi_E$ outside a rapidly null G_δ set.*

Here χ_E denotes the characteristic function of E. The measure of E is then defined as $\mu(E) = \mu(f)$, where $f = \chi_E$ almost everywhere as above. This is well-defined and locally well-behaved by the following results of X. Yu [Yu94].

Theorem 3.9 (WWKL)**.** *For $f, f' \in L^1(X,\mu)$, $||f - f'||_1 = 0$ if and only if $f = f'$ outside of a rapidly null G_δ set. If $f \leq f'$ outside of a rapidly null G_δ set, then $\mu(f) \leq \mu(f')$.*

For the rest of this paragraph, WWKL is assumed. Observe now that if $U \subseteq 2^\omega$ is open and if U is measurable in the above sense (that is, $\chi_U \in L^1(X,\mu)$), then we have $\mu(U) = \mu_I(U) = \mu(\chi_U)$. The last equality follows because if $U = \cup_{i<\omega}[p_i]$, the functions $\chi_{\cup_{i<n}[p_i]}$ are continuous and converge to χ_U in the L^1 norm. Finally, if A is a rapidly null G_δ set, then $\mu(\chi_A) = \mu_I(A) = 0$ because $\chi_A = 0$ outside of A itself. Therefore, when measurable characteristic functions for open or rapidly null G_δ sets exist, all our ways of defining measures for these sets coincide. The existence of a measurable characteristic function for an open set also guarantees that the measure of that open set exists in the model (and thus can be discussed directly, not just comparatively).

Finally, we will need to make use of some more explicit versions of known results from the literature. For example, we want to use Theorem 3.9, but as stated it does not give any bounds on the complexity of the rapidly null G_δ set. However, those bounds do exist and we need the uniformity that comes with them. So below we reprove several results in order to clarify the complexity of the null set of points that are being discarded. From here forward, we also restrict our attention to Cantor space with the fair coin measure, which is denoted by λ.

First, recall that if A_n is a sequence of rapidly null G_δ sets $A_n = \cap_i A_{n,i}$, the same trick used for producing a universal Martin-Löf test can also produce a rapidly null G_δ set $A \supseteq \cup_n A_n$. Just let $U_j = \cup_n A_{n,n+j+1}$, and let $A = \cap_j U_j$.

Much but not all of the rest of this section has been presented in [BGS02].

Proposition 3.10 (WWKL)**.** *Suppose that $\langle f_i \rangle$ is a sequence of ideal continuous functions of $C(X)$ which is rapidly Cauchy for the L^1 norm. Let*

$$A_n = \{x : \exists N \sum_{i=2n+1}^{N} |f_i(x) - f_{i+1}(x)| > 2^{-n}\}$$

Then $\mu(A_n) \leq 2^{-n}$.

Proof. Formally, A_n is a union of basic open sets $\cup_j [p_j]$ satisfying the condition. We can assume the $[p_j]$ are disjoint. By countable additivity, it suffices to show that $\mu(B) < 2^{-n}$ for all sets $B = \cup_{j<k} [p_j]$. Let N be large enough to witness that $[p_j] \subseteq A_n$ for all $j < k$. We have

$$2^{-n}\mu(B) = \int 2^{-n}\chi_B \leq \int \sum_{i=2n+1}^{N} |f_i - f_{i+1}| = \sum_{i=2n+1}^{N} \int |f_i - f_{i+1}| < 2^{-2n}$$

Thus $\mu(B) < 2^{-n}$, as needed. □

The corollaries use ACA_0 only to guarantee that a Cauchy sequence converges.

Corollary 3.11 (ACA_0)**.** *A name $\langle f_i \rangle$ for an element of $L^1(2^\omega)$ converges pointwise a.e. Furthermore, this pointwise convergence is achieved outside of the rapidly null G_δ set*

$$\bigcap_k \bigcup_{n>k} A_n$$

where A_n are defined as above.

Corollary 3.12 (WWKL)**.** *A name $\langle f_i \rangle$ for an element of $L^1(2^\omega)$ converges uniformly on each closed set*

$$B_k = 2^\omega \setminus \bigcup_{n \geq k} A_n,$$

where A_n are defined as above. Furthermore, the modulus of uniform convergence of f_i on B_k is primitive recursive: if $m > 2\max\{\ell, k\}$, then $|f_m(x) - f(x)| \leq 2^{-\ell}$.

Proof. Let $n = \max\{\ell, k\}$ and $x \in B_k$. Then $B_k \cap A_n = \emptyset$, and thus the series $f_m(x) + \sum_{i=m}^{\infty}(f_{i+1}(x) - f_i(x))$ converges absolutely, with

$$\sum_{i=m}^{\infty} |f_{i+1}(x) - f_i(x)| \leq \sum_{i=2n+1}^{\infty} |f_{i+1}(x) - f_i(x)| \leq 2^{-n} \leq 2^{-\ell}.$$

□

Corollary 3.13 (ACA_0)**.** *If $\langle f_i \rangle$ and $\langle g_i \rangle$ are two names for the same element of $L^1(2^\omega)$, then*

$$\lim_i f_i(x) = \lim_i g_i(x)$$

for almost all x. Furthermore, this pointwise convergence is achieved outside of a rapidly null G_δ set given by an explicit formula.

Proof. Let $A_n(f), A_n(g)$, and $A_n(f,g)$ be defined as in Proposition 3.10 applied to the rapidly Cauchy sequences $\langle f_i \rangle$, $\langle g_i \rangle$, and $\langle f_2, g_3, f_4, g_5, \dots \rangle$ respectively. Then the limits of $f_i(x)$ and $g_i(x)$ exist and agree for any x outside of three rapidly null G_δ sets. Combine these rapidly null G_δ sets into a single rapidly null G_δ set. □

Proposition 3.14 (ACA_0)**.** *If $\langle h_j \rangle$ is a sequence of functions of $L^1(2^\omega)$ rapidly converging to a function $g \in L^1(2^\omega)$, then*

$$\lim_{j\to\infty} h_j(x) = g(x)$$

for almost all x. Furthermore, this pointwise convergence is achieved outside of a rapidly null G_δ set given by an explicit formula.

Proof. Define $\langle f^i \rangle_{i\in\omega}$ by $f^i = h_i^{2i+1}$, where $\langle h_j^i \rangle_{i<\omega}$ is the given name for h_j. Then $\langle f^{i+2} \rangle_{i\in\omega}$ is rapidly Cauchy and is another name for g, which we can see because

$$\int |f^i - f^{i+1}| \le \int |f^i - h_i| + \int |h_i - h_{i+1}| + \int |h_{i+1} - f^{i+1}| \le 2^{-2i} + 2^{-i} + 2^{-2i}$$

and

$$\int |f^i - g| \le \int |f^i - h_i| + \int |h_i - g| \le 2^{-2i} + 2^{-i+1}.$$

Let $A_n(g)$ and $A_n(h_j)$ be the building blocks of infinitely many rapidly null G_δ sets as in Corollary 3.11, so that outside of these sets the notations $g(x)$ and $h_j(x)$ are well-defined as the pointwise limits of the given names for g and each h_j. Additionally, letting

$$C_k = \bigcup_{\substack{j>k \\ n>j}} A_n(h_j),$$

by Proposition 3.10, we have $\lambda(\cup_{n>j} A_n(h_j)) < 2^{-j}$ and thus $\lambda(C_k) < 2^{-k}$ and $\cap_k C_k$ is a rapidly null G_δ set. Combine into a single test

(1) the infinitely many rapidly null G_δ sets which result from applying Corollary 3.11 to the given names for g and each h_j
(2) the rapidly null G_δ set guaranteed by Corollary 3.13, so that for x outside of B, $\lim_i f^i(x) = g(x)$.
(3) $\cap_k C_k$.

By (1), if x avoids this test, then $h_j(x)$ and $g(x)$ are well-defined as the pointwise limit of the given names of g and h_j. By (2), if x avoids this test, then $\lim_i f^i(x) = g(x)$. Finally, we claim that if x avoids this test, then $\lim_j h_j(x) = \lim_i f^i(x)$. The limit on the right hand side exists, so it suffices to show that $\lim_j |f^j(x) - h_j(x)| = 0$. This follows by (3) because if $x \notin C_k$ for some k, then for all $j > k$ we have

$$|f^j(x) - h_j(x)| \le \sum_{i=2j+1}^{\infty} |h_j^i(x) - h_j^{i+1}(x)| \le 2^{-j}.$$

□

We have the following relationship between higher randomness and measure theory. This is surely known (and one could surely do better than Δ^1_1-random) but it is enough for our purposes.

Lemma 3.15. *Suppose that $f \in L^1(2^\omega)$, with name $\langle f^i \rangle_{i<\omega}$. Suppose that R is Δ^1_1-random relative to $\langle f^i \rangle_{i<\omega}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j<N} f(R^{[j]}) = \int_{2^\omega} f$$

Proof. Note that the randomness of R ensures that $f(R^{[j]})$ is well-defined as $\lim_i f^i(R)$. For any ε, we can find a measurable function $f_\varepsilon = \sum_{k=-\infty}^{\infty} k\varepsilon\chi_{A_k}$ where A_k are measurable sets which have Borel definitions uniformly in the name $\langle f^i \rangle$, and such that $|f(x) - f_\varepsilon(x)| < \varepsilon$ for all x outside of a G_δ set which also has a Borel definition relative to $\langle f^i \rangle$. Then the randomness of R ensures that the $R^{[j]}$ visit each A_k with the right limiting frequency, and that $|f(R^{[j]}) - k\varepsilon| < \varepsilon$ whenever $R^{[j]} \in A_k$. Thus $\frac{1}{N}\sum_{j<N} f(R^{[j]})$ is within ε of $\frac{1}{N}\sum_{j<N} f_\varepsilon(R^{[j]})$, and the latter tends to to $\int_{2^\omega} f_\varepsilon$ as N increases. Letting ε go to zero completes the proof. □

4. Regularity approximations and measure approximations

The following version of measurability for a set was implicit in [Yu93].

Definition 4.1. *A set B is* regularity-measurable *if there are G_δ sets A and C such that $A^c \subseteq B \subseteq C$ and $A \cap C$ is rapidly null.*

We bring up this definition because such a pair (A, C), which we could call a *regularity approximation* to B, would seem an obvious analog to the *Baire approximation* to a set B defined in [ADM+19]. We can use this notion of measurability to define the principle $\mathsf{CD\text{-}M}$ as follows.

Definition 4.2. *Let $\mathsf{CD\text{-}M}$ be the principle "Every completely determined Borel set is regularity-measurable".*

A difference between measure and category now arises. The Baire Category Theorem holds in RCA_0, so RCA_0 knows that the whole space is not meager. However, WWKL is needed in order to know that the whole space is not null.

Proposition 4.3. *Over RCA_0, $\neg\mathsf{WWKL}$ implies $\mathsf{CD\text{-}M}$.*

Proof. By Theorem 3.7, let A be an rapidly null G_δ set with empty complement. Let $C = 2^\omega$. Then for any set B, we have $A^c = \emptyset \subseteq B \subseteq C$, but $A \cap C$ is rapidly null because A is rapidly null. □

In the presence of WWKL, however, regularity-measurable coincides with the same notion of measurability given in Definition 3.8.

Proposition 4.4 (WWKL)**.** *Let $B \subseteq 2^\omega$ be any set. (Formally, the membership of B can be given by any formula in the language of second order arithmetic). Then B is regularity-measurable if and only if it is measurable in the sense of Definition 3.8.*

Proof. Suppose B is regularity-measurable. It follows that $A \cup C = 2^\omega$. Therefore, if $A = \cap_n A_n$ and $C = \cap_n C_n$, we have for each n that $A_n \cup C_n = 2^\omega$. Using WWKL, it follows that $\mu(A_n \cup C_n) = 1$, while $\mu(A_n \cap C_n) < 2^{-n}$ because $A \cap C$ is rapidly null. Define a sequence of functions $f_n : 2^\omega \to \{0,1\}$ and open sets B_n as follows. Given n, let s be large enough that $\mu(D_{n+1,s}) < 2^{-(n+1)}$, where we define

$$D_{n,s} = 2^\omega \setminus (A_{n,s} \cup C_{n,s}).$$

Let f_n be the characteristic function of $C_{n+1,s}$, and let

$$B_n = (A_{n+1} \cap C_{n+1}) \cup D_{n+1,s}.$$

Then $\mu(B_n) < 2^{-n}$. We have

$$||f_n - f_m||_1 = \mu(A_{n+1,s} \Delta A_{m+1,t})$$

where s and t are chosen as in the definition. Since $A_{n+1,s}\Delta A_{m+1,t} \subseteq B_n \cup B_m$, the sequence (f_n) is rapidly Cauchy and $f_n(x)$ converges to $\chi_B(x)$ for all x outside of $\cap_n(\cup_{k>n} B_k)$.

On the other hand, if B is measurable in the sense of Definition 3.8, then if $\langle f_n\rangle_{n\in\omega}$ is an L^1-name for χ_B, the sets $A_n = \{x : f_n(x) < 2/3\}$ and $C_n = \{x : f_n(x) > 1/3\}$ demonstrate that B is regularity-measurable. This follows because, letting $D = A_n \cap C_n$, we have

$$\frac{1}{3}\mu(D) = \int_D \frac{1}{3} \leq \int_D |f_n - \chi_B| \leq ||f_n - \chi_B||_1 \leq 2^{-n+1}.$$

□

The first step in evaluating the strength of $\mathsf{CD\text{-}M} + \mathsf{WWKL}$ is immediate.

Proposition 4.5. *Over* WWKL*, the statement "Every open subset of* 2^ω *is measurable" is equivalent to* ACA_0*.*

Proof. It is clear that ACA_0 proves the given statement. In the other direction, given an increasing sequence of real numbers (a_n) with each $a_n < 1$, let U be an open set designed so that $\mu_I(U) = \sup_n a_n$. For example, let U be the set which contains exactly those cylinders $[p^\frown 0]$ such that for some n, we have $.p^\frown 1 < a_n$, where $.p^\frown 1$ denotes the rational number with binary decimal expansion given by $p^\frown 1$. By WWKL, $\mu_I(U) = \mu(U)$. But $\mu(U)$ exists as a number, thus $\sup_n a_n$ exists. □

Combining Propositions 4.3 and 4.5, we arrive at the following curiosity. Let OSM be the statement "Every open set is regularity-measurable". Then by Proposition 4.5, we have that ACA_0 is equivalent to $\mathsf{WWKL} \wedge \mathsf{OSM}$, while Proposition 4.3 shows that RCA_0 proves $\mathsf{WWKL} \vee \mathsf{OSM}$. Thus we have a diamond formed of reasonably natural principles (though it must be admitted that OSM does not mean much outside of WWKL). We are not aware of any other diamond in reverse mathematics. By a diamond here we just mean informally an incomparable pair of principles A and B such that $A \wedge B$ is equivalent to some principle of interest, while $A \vee B$ follows from RCA_0.

We return now to our main discussion of the principle $\mathsf{CD\text{-}M}$. One direction of Proposition 4.5 can be extended to the Borel case as follows.

Proposition 4.6. *Over* WWKL*,* $\mathsf{CD\text{-}M}$ *implies* $\mathsf{L}_{\omega_1,\omega}\text{-}\mathsf{CA}$*.*

Proof. Given $\langle \phi_n\rangle_{n\in\omega}$ a sequence of completely determined formulas of $L_{\omega_1,\omega}$, turn them into Borel codes by changing their leaves as follows. If ϕ_n has `true` at a leaf, replace it with $[0^n1]$. If ϕ_n has `false` at a leaf, replace it with $\emptyset$. Now take the union of all of these codes. The resulting code is completely determined because each ϕ_n was completely determined and each $X \in 2^\omega$ belongs to at most one cylinder $[0^n 1]$. If f is a measurable characteristic function, then f is almost surely 1 on $[0^n 1]$ whenever ϕ_n is true, and almost surely 0 on $[0^n 1]$ whenever ϕ_n is false. Thus the sequence $\langle 2^n \int_{[0^n 1]} f \rangle_{n\in\omega}$ witnesses the satisfaction of $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$; this sequence assigns 1 to the true formulas and 0 to the false ones. □

The classical way of showing that every Borel set is measurable is to use arithmetic transfinite recursion to define a regularity approximation to $|T_\sigma|$ for each $\sigma \in T$. We present an effectivization of the classical proof which is particularly well-suited to our subsequent analysis.

Definition 4.7. *Let T be a code for a Borel set. A* measure decomposition *for T is a collection $\{f_\sigma : \sigma \in T, f_\sigma \in L^1(2^\omega)\}$ such that*

(1) If σ is a leaf, then f_σ is the characteristic function of $|T_\sigma|$.
(2) If σ is a union, then $f_\sigma = \sup_n f_{\sigma n}$.
(3) If σ is an intersection, then $f_\sigma = \inf_n f_{\sigma n}$.

All three equalities above refer to equality in the sense of the metric space $L^1(2^\omega)$. For example, the equation $f_\sigma = \sup_n f_{\sigma n}$ is shorthand for

$$\lim_{N\to\infty} \left(\sup_{n<N} f_{\sigma n} \right) = f_\sigma$$

and similarly for the other equation. In all cases, n ranges only over those numbers for which $\sigma n \in T$.

Proposition 4.8 (ACA_0)**.** *Suppose T is a code for a completely determined Borel set. If T has a measure decomposition, then $|T|$ is measurable.*

Proof. We need to show that $f_\emptyset$ is a.e. equal to the characteristic function of $|T|$. This is proved by arithmetic transfinite induction on T.

Observe that if we were willing to use Σ^1_2 transfinite induction and Σ^1_1-AC, the proof which inducts on the following statement would be very short: there is a rapidly null G_δ such that for all X outside of it, $f_\sigma(X) = 1$ if and only if $X \in |T_\sigma|$. Since we want to get away with arithmetic transfinite induction only, we need to identify the rapidly null G_δ in advance, then fix some X outside it, and then prove $f_\emptyset(X)$ is correct by transfinite induction on T.

We claim the following collection of rapidly null G_δ sets exists:

(1) For all σ, a rapidly null G_δ such that for all x outside of it, the name of f converges at x.
(2) For all leaf σ, a rapidly null G_δ set such that on its complement, f_σ is the characteristic function of $|T_\sigma|$
(3) For all union σ, a rapidly null G_δ set such that for all x in its complement, $f_\sigma(x) = \sup_n f_{\sigma n}(x)$
(4) For all intersection σ, same as the above except using $\inf_n f_{\sigma n}$.

The sets in (1) are obtained by uniform application of Corollary 3.11 to the given names for the functions f_σ. The sets in (2) are obtained by uniform application of Corollary 3.13 to f_σ and a standard name for the characteristic function of the clopen set $|T_\sigma|$. To obtain (3), use the fact that

$$\lim_{N\to\infty} \left(\sup_{n<N} f_{\sigma n} \right) = f_\sigma,$$

define $h_N = \sup_{n<N} f_{\sigma n}$, and find a sequence N_i such that $\langle h_{N_i}\rangle_{i\in\omega}$ is rapidly convergent to f_σ. Then apply Proposition 3.14 to $\langle h_{N_i}\rangle_{i\in\omega}$ together with the given name for f_σ. Although we have passed to a subsequence, because $h_N(x) \leq h_{N+1}(x)$ for all x, it follows that $h_N(x)$ converges if and only if $h_{N_i}(x)$ converges. (It will happen in our situation that $h_N(x)$ converges for all x, though we do not need this.) The procedure for (4) is similar.

Let A be a rapidly null G_δ set which contains all the bad-behavior sets above. Fix $X \notin A$. We claim that the map which sends σ to $f_\sigma(X)$ is an evaluation map for X in T. That is, we claim $f_\sigma(X) = 1$ if and only if $X \in |T_\sigma|$. The claim is proved by arithmetic transfinite induction on T. Observe that A contains all the points at which the proposed evaluation map fails to be right at the leaves or fails to satisfy the logic of the tree.

In particular, $f_\emptyset(X) = 1$ if and only if $X \in |T|$. □

Uniformly arithmetic in a sequence $\langle f_{\sigma n}\rangle_{n\in\omega}$, we may produce the functions $\sup_n f_{\sigma n}$ and $\inf_n f_{\sigma n}$. Therefore, ATR_0 suffices to create measure decompositions for all Borel sets. However, ACA_0 is enough to guarantee their uniqueness.

Proposition 4.9 (ACA_0)**.** *Suppose that T is a Borel code and $\langle f_\sigma\rangle_{\sigma\in T}$ and $\langle g_\sigma\rangle_{\sigma\in T}$ are two measure decompositions for T. Then for all $\sigma \in T$, $f_\sigma = g_\sigma$ as L^1 functions.*

Proof. By arithmetic transfinite induction. If for all n, $f_{\sigma n} = g_{\sigma n}$, then for all N, $\sup_{n<N} f_{\sigma n} = \sup_{n<N} g_{\sigma n}$. Therefore, these sequences have the same limit in the sense of L^1. □

Although we will show in the next section that $\mathsf{CD\text{-}M}$ is strictly weaker than ATR_0, the existence of measure decompositions is still necessary for $\mathsf{CD\text{-}M}$ to hold. Therefore, any model of $\mathsf{CD\text{-}M} + \neg\mathsf{ATR}_0$ will need some other way of producing measure decompositions.

Proposition 4.10 (ACA_0)**.** *If $\mathsf{CD\text{-}M}$ holds, then every completely determined Borel set has a measure decomposition.*

Proof. For any Borel code S, define an operation $S[n]$ as follows. Whenever a leaf of S is labeled by the clopen set $[p_0] \cup \cdots \cup [p_k]$, replace it with the clopen set $[0^n 1 p_0] \cup \cdots \cup [0^n 1 p_k]$. This has the effect of shrinking the set coded by S and relocating it to live completely inside the cone $[0^n 1]$.

Let $h : \omega \to T$ be a computable surjection. If T is completely determined, so is $\tilde{T}$, where

$$\tilde{T} = \cup_{n\in\omega} T_{h(n)}[n]$$

Colloquially, $\tilde{T}$ has been formed by taking each subtree T_σ of T and giving it its own dedicated part of the Cantor space. Now, if $\tilde{T}$ is measurable via the function $f \in L^1$, then the functions

$$f_\sigma(X) = f(0^{\min h^{-1}(\sigma)} 1 ^\frown X)$$

are a measure decomposition for T. □

5. Results

In this section we construct an ω-model $\mathcal{M}$ which satisfies $\mathsf{CD\text{-}M}$ but not ATR_0. Let R be a Π^1_1-random. Let $\mathcal{M}$ be the ω-model whose second-order part is $\bigcup_{i<\omega} \Delta^1_1(\bigoplus_{k<i} R^{[k]})$, where $R^{[k]}$ denotes the kth column of R.

Since the strings of $2^{<\omega}$ are in one-to-one correspondence with ω, we can assume such a correspondence is fixed and abuse notation to also let $G^{[p]}$ denote a column of G whenever $p \in 2^{<\omega}$ and $G \in 2^\omega$.

Proposition 5.1. *The model $\mathcal{M}$ does not satisfy ATR_0.*

Proof. Let a^* be a computable pseudo-ordinal. Then $a^* \in \mathcal{M}$. We claim that a^* has neither a descending sequence, nor a jump hierarchy, in $\mathcal{M}$. If $\Delta^1_1(R_0)$ had one, where $R_0 = \bigoplus_{k<i} R^{[k]}$, then by Theorem 2.1, $\omega_1^{R_0} = \omega_1^{ck}$. Thus there is an ordinal $b \in \mathcal{O}$ such that $H_b^{R_0}$ computes either a jump hierarchy on or a descending sequence in a^*. But recognizing a jump hierarchy or a descending sequence is arithmetic. So

$$\text{“}H_b^X \text{ computes a jump hierarchy or descending sequence for } a^*\text{”}$$

is a $\Sigma^0_{b+O(1)}$ statement, and it has measure either 0 or 1 because it describes a property of the tail of X. Because R_0 is sufficiently random, and satisfies the statement, the set has measure 1. But then any $b + O(1)$-generic also satisfies the statement. This is a contradiction because there are $b + O(1)$-generics in HYP, but a^* has no hyperarithmetic descending sequence nor any hyperarithmetic jump hierarchy. □

Proposition 5.2. *The model $\mathcal{M}$ satisfies $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$. Furthermore, whenever $R_0 \in M$ and $\langle \phi_i \rangle \in \Delta^1_1(R_0)$, if $\langle \phi_i \rangle$ is completely determined in $\mathcal{M}$, then it is completely determined in $\Delta^1_1(R_0)$.*

Proof. Suppose that $\langle \phi_j \rangle \in \Delta^1_1(\bigoplus_{i<k} R^{[i]})$ is a sequence of formulas of $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ which is completely determined in $\mathcal{M}$. Since $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ is a theory of hyperarithmetic analysis, it suffices to show that the sequence is determined in $\Delta^1_1(R_0)$, where $R_0 = \bigoplus_{i<k} R^{[i]}$. Fixing j, there is an $m > k$ such that $\Delta^1_1(\bigoplus_{i<m} R^{[i]})$ contains an evaluation map for ϕ_j. Let $R_1 = \bigoplus_{k \le i < m} R^{[i]}$. By Van Lambalgen's Theorem for Π^1_1-randoms, R_0 and R_1 are relatively Π^1_1-random. Since $\omega_1^{R_0 \oplus R_1} = \omega_1^{ck}$, there is some $a \in \mathcal{O}$ such that this evaluation map is computable from $H_a^{R_0 \oplus R_1}$. Then

$$C_j := \{X : H_a^{R_0 \oplus X} \text{ computes an evaluation map for } \phi_j\}$$

is a $\Delta^1_1(R_0)$ set which contains the $\Pi^1_1(R_0)$-random R_1. Therefore, C_j has measure 1, so any sufficiently random element computes an evaluation map for ϕ_j. Here, sufficiently random just means more random (relative to R_0) than the descriptive complexity of C_j. So there are elements of $\Delta^1_1(R_0)$ that are sufficiently random. Thus ϕ_j is determined in $\Delta^1_1(R_0)$. □

To show that $\mathcal{M}$ models CD-M, the following classical fact will be useful. It says roughly that if you approximate a bounded function f by using its average values on smaller and smaller partitions of the domain, the resulting sequence converges to f in the L^1 sense.

Lemma 5.3 (WWKL). *If $f \in L^1(2^\omega)$ is bounded and $h_i = \sum_{p \in 2^i} (2^i \int_{[p]} f)\chi_{[p]}$, then $h_i \to f$ in the L^1 norm.*

Proof. Given ε, use Corollary 3.12 to find a closed set B such that the restriction of f to B is continuous, and $\mu(2^\omega) - B < \varepsilon/M$, where M is a bound on f. Let i be large enough that on B, if $x \upharpoonright i = y \upharpoonright i$, then $|f(x) - f(y)| < \varepsilon$. Then for all strings $p \in 2^i$ and all $x_0 \in [p]$,

$$\begin{aligned} |h_i(x) - f(x)| &= |(2^i \textstyle\int_{[p]} f) - f(x_0)| \\ &= |(2^i \textstyle\int_{[p]\cap B} f) + (2^i \int_{[p]\setminus B} f) - 2^i \int_{[p]} f(x_0)| \quad (f(x_0) \text{ is a constant.}) \\ &\leq |2^i \textstyle\int_{[p]\cap B}(f - f(x_0))| + |2^i \int_{[p]\setminus B} f| + |2^i \int_{[p]\setminus B} f(x_0)| \\ &\leq 2^i \textstyle\int_{[p]\cap B} \varepsilon + 2(2^i \int_{[p]\setminus B} M) \end{aligned}$$

Therefore,

$$\begin{aligned} \int |h_i - f| &= \sum_{p\in 2^i} \int_{[p]} |h_i - f| \\ &\leq \sum_{p\in 2^i} \int_{[p]} 2^i (\textstyle\int_{[p\cap B]} \varepsilon + 2 \int_{[p]\setminus B} M) \\ &\leq \sum_{p\in 2^i} (\textstyle\int_{[p\cap B]} \varepsilon + 2 \int_{[p]\setminus B} M) \\ &= \textstyle\int_B \varepsilon + 2 \int_{2^\omega \setminus B} M \leq \varepsilon + 2\varepsilon. \end{aligned}$$

□

Lemma 5.4. *Suppose that $f \in L^1(2^\omega)$, with name $\langle f^i \rangle_{i<\omega}$. Suppose that R is Δ^1_1-random relative to $\langle f^i \rangle_{i<\omega}$. Define a sequence of functions g^i by*

$$g^i(X) = \lim_{N\to\infty} \frac{1}{N} \sum_{j<N} f((X \upharpoonright i)^\frown R^{[j]})$$

Then the functions are well-defined and $g^i \to f$ in the L^1 norm.

Proof. By 2^i-many applications of Lemma 3.15 to the functions $f_p(R) := f(p^\frown R)$, and since $2^i \int_{[p]} f = \int_{2^\omega} f_p$, we have $g^i(X) = \sum_{p\in 2^i} (2^i \int_{[p]} f)\chi_{[p]}$. Then $g^i \to f$ by Lemma 5.3. □

Theorem 5.5. *The principle* $\mathsf{CD\text{-}M}$ *is strictly weaker than* ATR_0*. In particular,* $\mathcal{M}$ *satisfies* $\mathsf{CD\text{-}M}$ *but not* ATR_0*.*

Proof. Suppose that we are given T, a completely determined Borel code. To simplify notation, we assume that $T \in \Delta^1_1$; the result for arbitrary $T \in \mathcal{M}$ follows by relativization. Let $R_0 = R^{[0]}$. Then abusing the column notation further, consider R_0 as being made out of infinitely many distinct and computably identifiable columns, one column for each pair (σ, j), where $\sigma \in \omega^{<\omega}, j \in \omega$, and let $R^{[\sigma,j]}$ denote the column allocated to that pair. Then letting

$$U := \{(p, \sigma, j) : p^\frown R_0^{[\sigma,j]} \in |T_\sigma|\}$$

we have $U \in \Delta^1_1(R_0)$ by Proposition 5.2. By the same reasoning, we also have that $U_\sigma := \{(p, j) : (\sigma, p, j) \in U\}$ satisfies $U_\sigma \in \Delta^1_1(R_0^{[\sigma]})$. where $R_0^{[\sigma]} = \bigoplus_{j<\omega} R_0^{[\sigma,j]}$.

Therefore, in $\Delta^1_1(R_0)$ we can also find the array of functions $\langle f^i_\sigma \rangle_{\sigma\in T, i\in\omega}$ defined as follows.

$$f^i_\sigma(X) := \limsup_{N\to\infty} \frac{1}{N} \sum_{j<N} U_\sigma(X \upharpoonright i, j)$$

Then define $f_\sigma = \limsup_i f^i_\sigma$. Since the functions $X \mapsto U_\sigma(X \upharpoonright i, j)$ are continuous and bounded above by 1, $f_\sigma \in L^1(2^\omega)$ by the monotone convergence theorem. Of course, the intention is to show that all limsups above can be replaced by limits a.e., and that f_σ represents $|T_\sigma|$ as a measurable set. We prove that $\langle f_\sigma \rangle$ is a measure decomposition by arithmetic transfinite induction within $\mathcal{M}$.

If σ is a leaf then the sequence of functions f^i_σ is eventually constant and equal to the characteristic function of the clopen set coded by T_σ, as desired.

So to complete the proof that $\langle f_\sigma \rangle_{\sigma \in T}$ is measure decomposition, it suffices to show that $\mathcal{M}$ models the following statement for each non-leaf $\sigma \in T$:

"If for all n, $\langle f_{\sigma n \tau} \rangle_{\tau \in T_{\sigma n}}$ is a measure decomposition, then $\langle f_{\sigma\tau} \rangle_{\tau \in T_\sigma}$ is a measure decomposition." That is, assuming $\mathcal{M}$ models the hypothesis, we need to show that $\mathcal{M}$ models:

(1) If σ is a union, then $f_\sigma = \sup_n f_{\sigma n}$
(2) If σ is an intersection, then $f_\sigma = \inf_n f_{\sigma n}$

We show the union case; the intersection case is completely symmetric. By Proposition 4.8, for each n, there is a rapidly null G_δ set such that on its complement, $f_{\sigma n}$ is the characteristic function of $|T_{\sigma n}|$. Inspecting the proof of Proposition 4.8, we see that the rapidly null G_δ sets guaranteed there have a uniform Δ^0_1 definition relative to the data $\langle f_{\sigma n \tau} : \tau \in T_{\sigma n}, n \in \omega \rangle$. Let A denote the rapidly null G_δ set obtained by combining these infinitely many tests into a single test. Define

$$R_0^{[<\sigma]} = \bigoplus_{\substack{\tau \in T_{\sigma n} \\ n \in \omega}} R_0^{[\sigma n \tau]}.$$

Since

$$A \leq_T \langle f_{\sigma n \tau} : \tau \in T_{\sigma n}, n \in \omega \rangle \leq_T \bigoplus_{\substack{\tau \in T_{\sigma n} \\ n \in \omega}} U_{\sigma n \tau} \in \Delta^1_1(R^{[<\sigma]})$$

and $R^{[\sigma]}$ is Δ^1_1-random relative to $R^{[<\sigma]}$, each column $R_0^{[\sigma,j]}$ avoids A. Therefore, for each $p \in 2^{<\omega}$ and each j and n, we have

$$p^\frown R_0^{[\sigma,j]} \in |T_{\sigma n}| \iff f_{\sigma n}(p^\frown R_0^{[\sigma,j]}) = 1.$$

Therefore,

$$\begin{aligned} (p,j) \in U_\sigma &\iff p^\frown R_0^{[\sigma,j]} \in |T_\sigma| \\ &\iff \exists n p^\frown R_0^{[\sigma,j]} \in |T_{\sigma n}| \\ &\iff \exists n f_{\sigma n}(p^\frown R_0^{[\sigma,j]}) = 1 \\ &\iff \sup_n f_{\sigma n}(p^\frown R_0^{[\sigma,j]}) = 1. \end{aligned}$$

Here $\sup_n f_{\sigma n}$ has a canonical L^1 name arithmetic in $\langle f_{\sigma n} : n \in \omega \rangle$, and the last bi-implication is justified by Proposition 3.14, since $p^\frown R_0^{[\sigma,j]}$ also avoids the rapidly null G_δ guaranteed there. Thus by Lemma 5.4,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j<N} \sup_n f_{\sigma n}(p^\frown R_0^{[\sigma,j]})$$

exists for all σ, and $\langle f^i_\sigma \rangle_{i \in \omega}$ is actually a name for $\sup_n f_{\sigma n}$. Therefore, by Proposition 3.14 and Corollary 3.13, for almost all x we have $f_\sigma(x) = \lim_i f^i_\sigma(x) =$

$\sup_n f_{\sigma n}(x)$. Theorem 3.9 then implies that $f_\sigma = \sup_n f_{\sigma n}$, which is what we wanted to prove. □

6. ω-models of CD-M are closed under Δ^1_1-randoms

In this section we show that any ω-model $\mathcal{M}$ of CD-M must be closed under Δ^1_1-randoms, in the sense that for every $Z \in \mathcal{M}$, there is an $R \in \mathcal{M}$ that is Δ^1_1-random relative to Z. We first review the machinery of decorating trees from [ADM+19]. All results summarized here relativize and they will be used in a relativized form, but we state them in unrelativized form to reduce clutter.

The purpose of the operation Decorate is to take a code for a Borel set which may not be completely determined, and force it to become determined for some "small" set of inputs, while not changing its membership facts for other inputs. In our case "small" will mean measure 0. Roughly speaking, we are going to make a code T and add decorations to ensure that all non Δ^1_1-randoms are determined in T. We will also make sure any measure decomposition is complicated enough to compute a Δ^1_1-random. That way, if there are no Δ^1_1-randoms then the tree is completely determined, at which point the existence of a computationally powerful measure decomposition leads to a contradiction.

Definition 6.1 ([ADM+19])**.** *A* nice decoration generator *is a partial computable function which maps any $b \in \mathcal{O}^*$ to alternating, b-ranked trees (P_b, N_b), where each P_b and N_b have an intersection or a leaf at their root.*

For example (and this is what we will use), there is a finite number k such that the following almost defines a nice decoration generator.

$$P_{b+k} = \{X : X \text{ is not } MLR^{H_b}, \text{ but for all } c <_* b,\, X \text{ is } MLR^{H_c}\} \cap \{X : X <_{\text{lex}} H_b\}$$

$$N_{b+k} = \{X : X \text{ is not } MLR^{H_b}, \text{ but for all } c <_* b,\, X \text{ is } MLR^{H_c}\} \cap \{X : X \geq_{\text{lex}} H_b\}$$

All that remains is to define P_b and N_b when b is within k successors of a limit ordinal; in that case we set both P_b and N_b to be b-ranked alternating codes for the empty set.

The operation Decorate is defined below using effective transfinite recursion (with parameter $<_*$ which is computable from $\emptyset'$), and therefore is well-defined on a-ranked trees T for all $a \in \mathcal{O}^{*,T}$.

Definition 6.2 ([ADM+19])**.** *The operation* Decorate *is defined as follows. The inputs are an a-ranked labeled tree T and a nice decoration generator h.*

$$\text{Decorate}(T,h) = \{\lambda\} \cup \bigcup_{\langle n\rangle \in T} \langle 2n\rangle^\frown \text{Decorate}(T_{\langle n\rangle}, h) \cup \bigcup_{b <_* \rho_T(\lambda)} \langle 2b+1\rangle^\frown \text{Decorate}(Q_b, h)$$

where $Q_b = P_b$ if λ is a $\cup$ in T, and $Q_b = N_b^c$ if λ is a $\cap$ in T.

The rank and label of λ in Decorate(T,h) *are defined to coincide with the rank and label of λ in T. The ranks and labels of the other nodes in* Decorate(T,h) *are inherited from* Decorate$(T_{\langle n\rangle}, h)$ *or* Decorate(Q_b, h) *as appropriate.*

If T is a-ranked, so is Decorate(T,h). Similarly, if T and each P_b and N_b are alternating, then Decorate(T,h) will also be alternating. (Note that in this case, N_b^c has a union at its root).

Lemma 6.3 ([ADM$^+$19])**.** *Let h be a nice decoration generator. Suppose $b \in \mathcal{O}$, and suppose that $X \notin |P_d| \cup |N_d|$ for any $d <_* b$. Then for any b-ranked tree T, $X \in |\operatorname{Decorate}(T, h)|$ if and only if $X \in |T|$.*

Lemma 6.4 ([ADM$^+$19])**.** *Let $a \in \mathcal{O}^*$ and $b \in \mathcal{O}$ with $b <_* a$. Let T be an alternating, a-ranked tree and let h be a nice decoration generator. Suppose $X \in |P_b| \cup |N_b|$. Then*

(1) X has a unique evaluation map in $\operatorname{Decorate}(T, h)$.
(2) This evaluation map is $H^{X \oplus T}_{b+O(1)}$-computable.

Theorem 6.5. *Suppose that $\mathcal{M}$ is an ω-model of* $\mathsf{WWKL} + \mathsf{CD\text{-}M}$*. Then for any $Z \in \mathcal{M}$, there is an $R \in \mathcal{M}$ such that R is Δ^1_1-random relative to Z.*

Proof. If $\mathcal{M}$ is a β-model, then $\mathcal{M}$ is already closed under Δ^1_1-randoms in the sense described above, because the statement $\exists R(R \text{ is } \Delta^1_1(Z)\text{-random})$ is a true $\Sigma^1_1(Z)$ statement, and any witness to its truth computes such an R.

On the other hand, if $\mathcal{M}$ is not a β-model, then there is a tree $S \in M$ such that $\mathcal{M}$ believes S to be well-founded, but in fact S is ill-founded. Without loss of generality, assume that $Z \geq_T S$; otherwise we end up with a Δ^1_1-random relative to $Z \oplus S$. There is a Z-computable procedure which, given any truly well-founded tree as input, produces an element of $\mathcal{O}^Z$ which bounds its rank. Apply this procedure to S to produce a pseudo-ordinal $a^* \in (\mathcal{O}^*)^Z$. Then $\mathcal{M}$ thinks that a^* is an ordinal. Let T be any Z-computable, alternating, (a^*+1)-ranked tree such that each level-one subtree T_n is a^*-ranked. We can assume T has a union at the root, though the symmetric choice would also work. Let h be the nice decoration generator which produces codes for P^Z_b and N^Z_b as follows (this is just the relativized form of what was defined above).

$$P^Z_{b+k} = \{X : X \text{ is not } MLR^{H^Z_b}, \text{ but for all } c <^Z_* b, X \text{ is } MLR^{H^Z_c}\} \cap \{X : X <_{\text{lex}} H^Z_b\}$$
$$N^Z_{b+k} = \{X : X \text{ is not } MLR^{H^Z_b}, \text{ but for all } c <^Z_* b, X \text{ is } MLR^{H^Z_c}\} \cap \{X : X \geq_{\text{lex}} H^Z_b\}$$

As above, we also define P^Z_b and N^Z_b to be b-ranked codes for the empty set in case b is within k successors of a limit ordinal. Now consider the tree $\operatorname{Decorate}^Z(T, h)$. Is it completely determined?

Suppose it is not completely determined; let X be an element that does not have an evaluation map. Since $\mathsf{CD\text{-}M} + \mathsf{WWKL}$ implies $\mathsf{L}_{\omega_1,\omega}\text{-}\mathsf{CA}$, every element of $HYP(X \oplus Z)$ is in $\mathcal{M}$. So by Lemma 6.4, for any $b \in \mathcal{O}^Z$, $X \notin |P^Z_b| \cup |N^Z_b|$ (if it were in this set, it would have a $HYP(X \oplus Z)$ evaluation map). But this means that X is Δ^1_1-random relative to Z, since each non-random belongs to some $|P^Z_b| \cup |N^Z_b|$.

So suppose that $\operatorname{Decorate}^Z(T, h)$ is completely determined. Then by $\mathsf{CD\text{-}M}$, it has a measure decomposition. We claim that any element R that is 1-random relative to the measure decomposition is in fact Δ^1_1-random relative to Z. It suffices to show that the measure decomposition computes H^Z_b for all $b \in \mathcal{O}^Z$. Fix $b \in \mathcal{O}^Z$ with $b <^Z_* a^*$ and observe that $\operatorname{Decorate}^Z(P_{b+k}, h)$ appears as a level-one subtree of $\operatorname{Decorate}^Z(T, h)$. Thus, by examining the definition of P_{b+k}, which has an intersection at the root and $\{X : X <_{\text{lex}} H^Z_b\}$ as a level-one subtree, we see that $\operatorname{Decorate}^Z(\{X : X <_{\text{lex}} H^Z_b\}, h)$ appears as a level-two subtree of $\operatorname{Decorate}^Z(T, h)$. (Here of course, $\{X : X <_{\text{lex}} H^Z_b\}$ is represented using an approximately b-ranked formula of $L_{\omega_1,\omega}$, but this formula contributes computational, not topological, complexity.) Therefore, there is an L^1 function f included in the measure decomposition which is equal to the characteristic function of $\text{Decorate}^Z(\{X : X <_{\text{lex}} H_b^Z\}, h)$ almost everywhere. We claim that $\int f = H_b^Z$, where here we regard H_b^Z as a number in $[0, 1]$ given by its binary expansion. Using WWKL, it suffices to provide another L^1 function g which has $\int g = H_b^Z$ and such that g is equal to the characteristic function of $\text{Decorate}^Z(\{X : X <_{\text{lex}} H_b^Z\}, h)$ almost everywhere. Let g be the canonical measurable characteristic function of the open set $\cup_{p <_{\text{lex}} H_b^Z}[p]$. Then by Lemma 6.3, for any X that is $MLR^{H_b^Z}$, since $X \notin |P_d^Z| \cup |N_d^Z|$ for any $d <_*^Z b + k$, we have $X \in \text{Decorate}^Z(\{X : X <_{\text{lex}} H_b^Z\}, h)$ if and only if $X <_{\text{lex}} H_b^Z$, which is true if and only if $g(X) = 1$. This completes the proof. □

References

[ADM+19] Eric P. Astor, Damir Dzhafarov, Antonio Montalbán, Reed Solomon, and Linda Brown Westrick. The determined property of Baire in reverse math. *J. Symbolic Logic*, Accepted 2019. Available arXiv:1809.03940.

[ADR12] Jeremy Avigad, Edward T. Dean, and Jason Rute. Algorithmic randomness, reverse mathematics, and the dominated convergence theorem. *Ann. Pure Appl. Logic*, 163(12):1854–1864, 2012.

[BGS02] Douglas K. Brown, Mariagnese Giusto, and Stephen G. Simpson. Vitali's theorem and WWKL. *Arch. Math. Logic*, 41(2):191–206, 2002.

[Bog07] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.

[CNY08] C. T. Chong, Andre Nies, and Liang Yu. Lowness of higher randomness notions. *Israel J. Math.*, 166:39–60, 2008.

[DFSW17] Damir Dzhafarov, Stephen Flood, Reed Solomon, and Linda Brown Westrick. Effectiveness for the Dual Ramsey Theorem. Submitted 2017. Available arXiv:1710.00070.

[DM13] Adam R. Day and Joseph S. Miller. Randomness for non-computable measures. *Trans. Amer. Math. Soc.*, 365(7):3575–3591, 2013.

[HN07] Greg Hjorth and André Nies. Randomness via effective descriptive set theory. *J. Lond. Math. Soc. (2)*, 75(2):495–508, 2007.

[Sim09] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edition, 2009.

[Ste73] Jacques Stern. Réels aléatoires et ensembles de mesure nulle en théorie descriptive des ensembles. *C. R. Acad. Sci. Paris Sér. A-B*, 276:A1249–A1252, 1973.

[Ste75] Jacques Stern. Some measure theoretic results in effective descriptive set theory. *Israel J. Math.*, 20(2):97–110, 1975.

[YS90] Xiaokang Yu and Stephen G. Simpson. Measure theory and weak König's lemma. *Arch. Math. Logic*, 30(3):171–180, 1990.

[Yu90] Xiaokang Yu. Radon-Nikodým theorem is equivalent to arithmetical comprehension. In *Logic and computation (Pittsburgh, PA, 1987)*, volume 106 of *Contemp. Math.*, pages 289–297. Amer. Math. Soc., Providence, RI, 1990.

[Yu93] Xiaokang Yu. Riesz representation theorem, Borel measures and subsystems of second-order arithmetic. *Ann. Pure Appl. Logic*, 59(1):65–78, 1993.

[Yu94] Xiaokang Yu. Lebesgue convergence theorems and reverse mathematics. *Math. Logic Quart.*, 40(1):1–13, 1994.

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