The Dual Ramsey Theorem and the Property of Baire

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The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let $[\omega]^k$ denote the set of all k-element subsets of ω.

Theorem (Ramsey's Theorem)

If $[\omega]^k = \cup_{i<l} C_i$, there is $H \subseteq \omega$ such that $[H]^k \subseteq C_i$ for some i.

Instead of k-element subsets of ω, we consider partitions of ω into k pieces. Notation:

- $(\omega)^k$ is the set of partitions of ω into exactly k pieces.
- $(\omega)^\omega$ is the set of partitions of ω into infinitely many pieces.
- If $x \in (\omega)^\omega$ and y is coarser than x, we write $y \in (x)^\omega$ (in case y is infinite) or $y \in (x)^k$ (if y has k blocks.)

Theorem (Dual Ramsey Theorem, Carlson & Simpson 1986)

If $(\omega)^k = \cup_{i<l} C_i$ is Borel, there is $x \in (\omega)^\omega$ such that $(x)^k \subseteq C_i$ for some i.

What's Known

We write DRT^k_l for the Dual Ramsey Theorem for k partitions and l colors.

Background knowledge:

- As usual, applying DRT^k_2 repeatedly yields DRT^k_l.
- Open-DRT^{k+1}_l computably implies RT^k_l. (Miller & Solomon 2004)
- For $k \geq 4$, Open-$DRT^k_l \to ACA_0$ over RCA_0. (Miller & Solomon 2004).
- Miller & Solomon 2004 and Erhard 2013: various results related to the Carlson-Simpson Lemma, which is the combinatorial core of the DRT.

Our goal: Understand the topological aspects of the DRT.

This is joint work with Damir Dzhafarov, Stephen Flood and Reed Solomon.

The 20 second version of this talk

The only effect fancy topology has on DRT^{3+} is
making the comeager approximation to the coloring hard to find.

On the other hand,
fancy topology is the only way to give DRT^2 content.

The Strength of Topologically Clopen DRT^{3+}

Theorem (Dzhafarov, Flood, Solomon, W.)

Let $k \geq 3$. For each computable ordinal α, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

Proof:

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \cup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given α, let f be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
- (This means $f \leq_T \emptyset^{(\alpha)}$, and for every g which dominates f, $\emptyset^{(\alpha)} \leq_T g$.)
- Let p be Red if $f(a_p) < b_p$, and Blue otherwise.
- Let $x \in (\omega)^\omega$ be an infinite homogeneous partition, $x = \{X_0, X_1, \ldots\}$.
- Then x is homogeneous for Red; for sufficiently large M, consider its coarsening $p = \{X_0, \bigcup_{i=1}^{M-1} X_i, \bigcup_{i=M}^{\infty} X_i\}$
- Then $g(n) := \min X_n$, and g dominates f.

Criticism of the theorem

This theorem doesn't use the interesting pieces of the DRT.

- The coloring it produces is topologically clopen.
- It uses no combinatorics, only growth rate.

What this theorem tells us about topology in the DRT

If one wanted to consider topologically interesting Borel colorings of $(\omega)^k$, how would those colorings be represented?

- A well-founded Borel code would seem the default.
- But, a $\emptyset^{(\alpha)}$-computable clopen coloring has a computable $\sim \Delta_\alpha$ code.
- If we allow well-founded Borel codes to represent topology, the coloring of the previous theorem can't be avoided.
- It uses fake topological complexity to hide its Δ_α information.
- In this example, DRT^{3+} could be seen as a strange way to realize the statement "every Borel set has the property of Baire"

The anatomy of the Carlson-Simpson proof

Carlson and Simpson prove the DRT as follows.

- Define a variation of DRT^k called DRT^k_A.
- Given an instance of DRT^k, cook up a set X via ω-many nested applications of various instances of DRT^{k-1}_A
- Applying the Carlson-Simpson Lemma (combinatorial lemma) to X gives the desired homogeneous partition.
- As a base case, to solve an instance of DRT^0_A, start with a comeager approximation to the given coloring and compute a solution from it.

How to prevent coding from masquerading as topology

Idea: Require a Δ_α coloring to also come equipped with a comeager approximation.

(That is, when

$$(\omega)^k = \bigcup_{i<l} C_i, \qquad C_i \text{ is } \Delta_\alpha$$

insist that along with a Δ_α code for the C_i, one is provided with Σ_1 codes for open sets U_i and D_n such that $\bigcup_{i<l} U_i$ is dense, each D_n is dense and

$$C_i = U_i \text{ on } \cap_n D_n.)$$

We will see that in fact, the behavior of the coloring on a meager set is irrelevant.

An Alternate Proof of the DRT

Definition

A coloring of $(\omega)^k$ is *reduced* if for $p \in (\omega)^k$, the color of p depends only on:

- The least element a of the kth block of p
- All block membership information for all elements $n < a$.

Reduced colorings are clopen.

Theorem (DFSW)

Let $(\omega)^k = \cup_{i<l} C_i$ be any coloring that satisfies the property of Baire. Uniformly in a comeager approximation to $\cup_i C_i$, there is a reduced coloring of $(\omega)^k$ such that any set homogeneous for it computes (together with the comeager approximation) a homogeneous solution to the original.

So, Borel-DRT is reducible to Open-DRT if we rule out coding via the Property of Baire.

An Alternate Proof of the DRT

Definition

A coloring of $(\omega)^k$ is *reduced* if for $p \in (\omega)^k$, the color of p depends only on:

- The least element a of the kth block of p
- All block membership information for all elements $n < a$.

Let $k_{fin}^{<\omega}$ be the set of all finite strings σ on $\{0, \ldots, k-1\}$ such that every symbol appears in σ at least once, and the first appearance of i precedes the first appearance of $i+1$.

The Combinatorial Dual Ramsey Theorem is the DRT for reduced colorings.

Theorem (Combinatorial Dual Ramsey Theorem ($cDRT$))

Let $(k-1)_{fin}^{<\omega} = \cup_{i<l} C_i$ be a coloring. Then there is $x \in (\omega)^\omega$ such that for every $p \in (x)^k$, $p \restriction k_p \in C_i$ for some i, where k_p is the first element of the kth block of p.

The Carlson-Simpson Lemma

Theorem (Combinatorial Dual Ramsey Theorem ($cDRT$))

Let $(k-1)^{<\omega}_{fin} = \cup_{i<l} C_i$ be a coloring. Then there is $x \in (\omega)^\omega$ such that for every $p \in (x)^k$, $p \in C_i$ for some i.

Lemma (Carlson-Simpson Lemma)

Let $(k-1)^{<\omega}_{fin} = \cup_{i<l} C_i$ be a coloring. Then there is $x \in (\omega)^\omega$ such that for every $p \in (x)^k$ **which keeps the first $(k-1)$ blocks of x separated**, *$p \in C_i$ for some i.*

An Alternate Proof of the DRT

An alternate proof of the DRT:

- Given an instance of DRT^k, apply the Property of Baire to get a comeager approximation.
- Using the comeager approximation, pass to an instance of $cDRT^k$.
- Define a variation of $cDRT^k$ called CSL^k (Carlson-Simpson Lemma).
- Given an instance of $cDRT^k$, cook up a set X via ω-many nested applications of various instances of CSL^{k-1}
- The result X is an instance of $cDRT^{k-1}$.
- The base case is computably true.

Implications

Thus, Borel-DRT may be cleanly cleaved into two disparate steps:

- Every Borel set has the Property of Baire
- Combinatorial Dual Ramsey Theorem

Corollary

The Dual Ramsey Theorem holds for any coloring that has the Property of Baire.

(This possibility was mentioned but not pursued in Carlson & Simpson 1986.)

Open Questions

How strong is $cDRT$? (Reverse-math, computable-analysis, descriptive strength.)

Is the Carlson-Simpson Lemma strictly weaker than $cDRT$?

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The Weakness of DRT^2

Theorem (DFSW)

Open-DRT^2 is computably true.

Proof:

- Pass computably to $cDRT^2$.
- It colors strings of the form 0^n, so it just colors numbers.
- Some color is used infinitely often, let's say Blue.
- A homogeneous partition is $\{n : 0^n \text{ not Blue}\}, \{n_1\}, \{n_2\}, \ldots$.

Similarly, if the coloring is given as a comeager approximation, it computes a homogeneous set.

The Weakness of DRT^2

The principle DRT^2 is so weak that unlike DRT^{3+}, it (probably) cannot preserve the computational complexity of its input.

Theorem (DFSW)

$cDRT_2^2$ for Δ_α-coded colorings is computably uniformly equivalent to the statement that for every Δ_α-coded subset of ω, there is an infinite set contained in either it or its complement.

Proof. Suppose we have a Δ_α subset $A \subseteq \omega$.

- This could be considered as a 2-coloring for $cDRT^2$.
- An infinite subset of A or $\overline{A}$ computes a homogeneous partition.
- An infinite homogeneous partition computes an infinite subset of A or $\bar{A}$. If the partition is $x = \{X_0, X_1, \dots\}$, the subset is $\{\min X_1, \min X_2, \dots\}$

In general, an infinite subset of A or $\overline{A}$ computes nothing in particular; it could certainly fail to compute A.

Leaving the Property of Baire in DRT^2_2

So, using Δ_α codes for a coloring for DRT^2 does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:

- $cDRT^2$ for Δ_α-coded colorings (of ω)
- DRT^2 for Δ_α-coded colorings (of $(\omega)^2$)

Open questions:

- Is the second principle strictly stronger than the first?
- What if we have a Δ_α code for a coloring of $(\omega)^2$ which we know is clopen?