EFFECTIVENESS OF HINDMAN'S THEOREM FOR BOUNDED SUMS

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This paper is dedicated to Rod Downey in honor of his outstanding contributions to computability theory and his leadership role in mentoring and exposition. Two of the authors had the pleasure of being mentored by Rod as postdocs.

Abstract. We consider the strength and effective content of restricted versions of Hindman's Theorem in which the number of colors is specified and the length of the sums has a specified finite bound. Let $\mathsf{HT}^{\leq n}_k$ denote the assertion that for each k-coloring c of $\mathbb{N}$ there is an infinite set $X \subseteq \mathbb{N}$ such that all sums $\sum_{x \in F} x$ for $F \subseteq X$ and $0 < |F| \leq n$ have the same color. We prove that there is a computable 2-coloring c of $\mathbb{N}$ such that there is no infinite computable set X such that all nonempty sums of at most 2 elements of X have the same color. It follows that $\mathsf{HT}^{\leq 2}_2$ is not provable in RCA_0 and in fact we show that it implies SRT^2_2 in $\mathsf{RCA}_0 + \mathsf{B}\Pi^0_1$. We also show that there is a computable instance of $\mathsf{HT}^{\leq 3}_3$ with all solutions computing $0'$. The proof of this result shows that $\mathsf{HT}^{\leq 3}_3$ implies ACA_0 in RCA_0.

1. Introduction

Hindman's Theorem (denoted HT) asserts that for every coloring of $\mathbb{N}$ with finitely many colors there is an infinite set $X \subseteq \mathbb{N}$ such that all nonempty finite sums of distinct elements of X have the same color. Hindman's Theorem was proved by Neil Hindman [6]. Hindman's original proof was a complicated combinatorial argument, and

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simpler proofs have been subsequently found. These include combinatorial proofs by Baumgartner [1] and by Towsner [12] and a proof using ultrafilters by Galvin and Glazer (see [4]).

We assume that the reader is familiar with the basic concepts of computability theory and of reverse mathematics. For information on these topics see, respectively, the books by Soare [11] and Simpson [10]. Our notation is standard. In particular, let $\mathbb{N}$ be the set of positive integers, and for $k \in \mathbb{N}$ we identify k and $\{0, 1, \ldots, k-1\}$. A *k-coloring* of $\mathbb{N}$ is a function $c : \mathbb{N} \to k$. A set $Z \subseteq \mathbb{N}$ is *monochromatic* for a coloring c if $c(x) = c(y)$ for all $x, y \in Z$.

The effective content of Hindman's Theorem and its strength as a sentence of second-order arithmetic were studied by Blass, Hirst, and Simpson [2]. They showed that every computable instance c of HT has a solution X computable from $0^{(\omega+1)}$ and, correspondingly, that HT is provable in the system ACA_0^+ obtained by adding to RCA_0 the statement $(\forall X)[X^{(\omega)}$ exists]. In the other direction, they showed that there is a computable instance c of HT such that all solutions X compute $0'$ and, correspondingly, that HT implies ACA_0 in RCA_0.

There is obviously a significant gap between the upper and lower bounds given in the previous paragraph, and closing these gaps has been a major issue in reverse mathematics. In particular it is not known whether there is an n such that every computable instance of Hindman's Theorem has a Σ^0_n solution and, correspondingly, whether HT is provable from ACA_0 in RCA_0.

In the current paper we study the strength and effective content of Hindman's Theorem when it is restricted to sums of bounded length. One might think that such restricted versions of Hindman's Theorem are far weaker than Hindman's Theorem itself, but in fact it is unknown whether this is true. In fact it is a major open problem in combinatorics (see [7], Question 12) whether every proof of Hindman's Theorem for sums of length at most two also proves Hindman's Theorem. We now state these bounded versions formally.

Definition 1.1. For a finite nonempty set $F \subseteq \mathbb{N}$, we let $\sum F$ denote the sum of the elements of F. For $X \subseteq \mathbb{N}$ and $n \geq 1$, we define

$$\mathrm{FS}^{\leq n}(X) = \left\{ \sum F \mid F \subseteq X \text{ and } 1 \leq |F| \leq n \right\}.$$

Definition 1.2. Let $\mathsf{HT}^{\leq n}_k$ denote the statement that for every coloring $c : \mathbb{N} \to k$, there is an infinite set X such that $\mathrm{FS}^{\leq n}(X)$ is monochromatic.

We show in Section 2 that for every Δ^0_2 set X there is a computable instance c of $\mathsf{HT}^{\leq 2}_2$ such that every solution H to c computes an infinite subset of X or $\overline{X}$. It follows that $\mathsf{HT}^{\leq 2}_2$ has a computable instance with no computable solution and hence is not provable in RCA_0. In fact, our proof shows that $\mathsf{HT}^{\leq 2}_2$ implies SRT^2_2 (Stable Ramsey's Theorem for 2-colorings of pairs) in $\mathsf{RCA}_0 + \mathsf{B}\Pi^0_1$, where $\mathsf{B}\Pi^0_1$ is the bounding principle for Π^0_1 formulas. Next we show in Section 3 that there is a computable instance of $\mathsf{HT}^{\leq 3}_3$ such that every solution computes $0'$ and, correspondingly, that $\mathsf{HT}^{\leq 3}_3$ implies ACA_0 in RCA_0. Our proof uses a very ingenious trick from Blass, Hirst, and Simpson [2], combined with some new ideas.

The final section lists many open questions.

2. Hindman's Theorem for sums of length at most 2

Our first theorem concerns $\mathsf{HT}^{\leq 2}_2$ and implies that it has a computable instance c with no computable solution X.

Theorem 2.1. *Let A be a Δ^0_2 set. There is a computable coloring $c : \mathbb{N} \to 2$ such that if W is an infinite set with $FS^{\leq 2}(W)$ monochromatic, then there is an infinite set $Y \leq_T W$ such that $Y \subseteq A$ or $Y \subseteq \overline{A}$.*

Proof. Fix a Δ^0_2 set A and a computable $\{0,1\}$-valued function $f(k,s)$ such that $A(k) = \lim_s f(k,s)$. For $k \geq 0$ and $i \in \{1,2\}$, define

$$\mathcal{O}_{k,i} = \{s \in \mathbb{N} \mid s \equiv i \cdot 3^k \bmod 3^{k+1}\}.$$

If s is written as $s = i_0 \cdot 3^{k_0} + \cdots + i_m \cdot 3^{k_m}$ with $k_0 < \cdots < k_m$ and each $i_j \in \{1,2\}$, then $s \in \mathcal{O}_{k,i}$ if and only if $k = k_0$ and $i = i_0$. The sets $\mathcal{O}_{k,i}$ give a computable partition of $\mathbb{N}$ such that if $s,t \in \mathcal{O}_{k,1}$, then $s+t \in \mathcal{O}_{k,2}$ and if $s,t \in \mathcal{O}_{k,2}$, then $s+t \in \mathcal{O}_{k,1}$. Furthermore, if $s \in \mathcal{O}_{k,i}$ and $t \in \mathcal{O}_{k',i'}$ with $k < k'$ and $i' \in \{1,2\}$, then $s+t \in \mathcal{O}_{k,i}$. For any $s \in \mathbb{N}$, we let k_s, i_s be the unique numbers k, i such that $s \in \mathcal{O}_{k,i}$. We define our coloring c by

$$c(s) = \begin{cases} f(k_s, s) & \text{if } i_s = 1, \\ 1 - f(k_s, s) & \text{if } i_s = 2. \end{cases}$$

The first important property of this coloring is that for each k we have $c(s) \neq c(t)$ whenever $s \in \mathcal{O}_{k,1}$ and $t \in \mathcal{O}_{k,2}$ are both sufficiently large. This holds since for sufficiently large $s \in \mathcal{O}_{k,1}$ and $t \in \mathcal{O}_{k,2}$ we have $c(s) = f(k,s) = A(k)$ and $c(t) = 1 - f(k,t) = 1 - A(k)$. It follows that for any monochromatic set Z, either $Z \cap \mathcal{O}_{k,1}$ is finite or $Z \cap \mathcal{O}_{k,2}$ is finite.

Fix an infinite set W with $\mathrm{FS}^{\leq 2}(W)$ monochromatic. We claim that $W \cap \mathcal{O}_{k,i}$ is finite for each $k \in \mathbb{N}$ and $i \in \{1,2\}$. Suppose first that $W \cap \mathcal{O}_{k,1}$ is infinite. Let S be the set of all sums $a+b$ where a, b are distinct elements of $W \cap \mathcal{O}_{k,1}$. Then S is infinite and $S \subseteq \mathcal{O}_{k,2} \cap \mathrm{FS}^{\leq 2}(W)$. Let $Z = W \cup S$. Then Z is monochromatic since $Z \subseteq \mathrm{FS}^{\leq 2}(W)$. Furthermore, $Z \cap \mathcal{O}_{k,1}$ and $Z \cap \mathcal{O}_{k,2}$ are both infinite, contradicting the previous paragraph. This shows that $W \cap \mathcal{O}_{k,1}$ is finite, and the proof that $W \cap \mathcal{O}_{k,2}$ is finite is analogous. It follows that there are infinitely many k such that $W \cap (\mathcal{O}_{k,1} \cup \mathcal{O}_{k,2})$ is nonempty. We call such numbers k *informative* since, as the next claim shows, W can compute $A(k)$ for all informative k.

We claim that if $s \in W \cap (\mathcal{O}_{k,1} \cup \mathcal{O}_{k,2})$ then $f(k,s) = A(k)$. To prove this claim, assume first that $s \in W \cap \mathcal{O}_{k,1}$. Note that $\mathrm{FS}^{\leq 2}(W) \cap \mathcal{O}_{k,1}$ is infinite, since it contains all sums $s + b$ with $b \in W \cap \mathcal{O}_{k',i'}$ for some $k' > k$, and $i' \in \{1,2\}$, and there are infinitely many such b. Let t be an element of $\mathrm{FS}^{\leq 2}(W) \cap \mathcal{O}_{k,1}$ sufficiently large that $f(k,t) = A(k)$. Since $\mathrm{FS}^{\leq 2}(W)$ is monochromatic, $c(s) = c(t)$. Hence $f(k,s) = c(s) = c(t) = f(k,t) = A(k)$. The proof for $s \in W \cap \mathcal{O}_{k,2}$ is analogous. The claim is proved.

For $i \in \{0,1\}$ let B_i be the set of numbers k such that W can compute that $A(k) = i$. More precisely, define

$$B_i = \{k \mid (\exists s)[s \in W \cap (\mathcal{O}_{k,1} \cup \mathcal{O}_{k,2}) \;\&\; f(k,s) = i]\}$$

By the above claim, $B_1 \subseteq A$ and $B_0 \subseteq \overline{A}$. Also, each set B_i is c.e. in W. Finally, if k is informative, then $k \in B_0 \cup B_1$. Since there are infinitely many informative numbers, $B_0 \cup B_1$ is infinite, and so B_0 or B_1 is infinite. Fix i such that B_i is infinite, and let Y be an infinite W-computable subset of B_i. Then Y is the desired infinite W-computable subset of A or $\overline{A}$. □

The next corollary follows by taking A to be a bi-immune Δ^0_2 set, for example a Δ^0_2 1-generic set.

Corollary 2.2. *There is a computable coloring $c : \mathbb{N} \to 2$ such that if X is an infinite computable set, then $FS^{\leq 2}(X)$ is not monochromatic.*

The next corollary follows immediately.

Corollary 2.3. $\mathsf{HT}^{\leq 2}_2$ *is not provable in* RCA_0.

We now sharpen the previous corollary. Let SRT^2_2 be Stable Ramsey's Theorem for 2-colorings of pairs as defined in Statement 7.5 of [5].

Corollary 2.4. $\mathsf{RCA}_0 + \mathsf{B}\Pi^0_1 \vdash \mathsf{HT}^{\leq 2}_2 \to \mathsf{SRT}^2_2$.

To prove the corollary, first let D^2_2 be the assertion that for every $\{0,1\}$-valued function $f(x,s)$ such that for all x, $\lim_s f(x,s)$ exists there is an infinite set G and $j<2$ such that $\lim_s f(x,s)=j$ for all $x\in G$. (The principle D^2_2 was defined in Statement 7.8 of [5].) Formalizing the proof of the theorem shows that $\mathsf{HT}^{\leq 2}_2$ implies the principle D^2_2 in $\mathsf{RCA}_0+\mathsf{B}\Pi^0_1$. (We thank Denis Hirschfeldt for pointing out to us that $\mathsf{B}\Pi^0_1$ is apparently needed to show in the proof of Theorem 2.1 that there are infinitely many k such that $W\cap(\mathcal{O}_{k,1}\cup\mathcal{O}_{k,2})$ is nonempty from the facts that W is infinite and has finite intersection with each $\mathcal{O}_{k,i}$.) Then SRT^2_2 follows from $\mathsf{D}^2_2+\mathsf{B}\Pi^0_1$ by the proof of Lemma 7.10 of [5]. (The latter proof contains a hidden use of hidden use of $\mathsf{B}\Pi^0_1$.) We do not know whether the use of $\mathsf{B}\Pi^0_1$ in this corollary is necessary, though it can be eliminated from the proof that D^2_2 implies SRT^2_2 by Theorem 1.4 of Chong, Lempp, and Yang [3].

3. Hindman's Theorem for sums of length at most 3

We now strengthen the results of the previous section, at the cost of allowing longer sums and more colors. We start by considering $\mathsf{HT}^{\leq 3}_4$ and then improve the results to $\mathsf{HT}^{\leq 3}_3$.

Theorem 3.1. *There is a computable coloring $c:\mathbb{N}\to 4$ such that if X is infinite with $FS^{\leq 3}(X)$ monochromatic, then $0'\leq_T X$.*

Proof. Let $f:\mathbb{N}\to\mathbb{N}$ be a computable 1-1 function. We will define a computable coloring $c:\mathbb{N}\to 4$ such that if X is infinite with $\mathrm{FS}^{\leq 3}(X)$ monochromatic, then X computes range(f).

For $n\in\mathbb{N}$, write $n=i_0\cdot 3^{k_0}+\cdots+i_\ell\cdot 3^{k_\ell}$ with $k_0<\cdots<k_\ell$ and each $i_j\in\{1,2\}$. Define $\lambda(n)=k_0$, $\mu(n)=k_\ell$ and $i(n)=i_0$. We will use several properties of the functions $\lambda(n)$, $\mu(n)$ and $i(n)$. The following are all straightforward to establish.

(P1) If $\lambda(n)<\lambda(m)$, then $\lambda(n+m)=\lambda(n)$ and $i(n+m)=i(n)$.
(P2) If $\lambda(n)=\lambda(m)$ and $i(n)=i(m)=1$, then $\lambda(n+m)=\lambda(n)$ and $i(n+m)=2$.
(P3) If $\lambda(n)=\lambda(m)$ and $i(n)=i(m)=2$, then $\lambda(n+m)=\lambda(n)$ and $i(n+m)=1$.
(P4) If $\mu(n)<\lambda(m)$, then $\lambda(n+m)=\lambda(n)$ and $\mu(n+m)=\mu(m)$.

For $n=i_0\cdot 3^{k_0}+\cdots+i_\ell\cdot 3^{k_\ell}$ with the i_j and k_j as above, we refer to the intervals (k_j,k_{j+1}) for $j<\ell$ as the *gaps of n*. A gap (a,b) of n is a *short gap in n* if there is a $y\leq a$ such that $y\in$ range(f) but there is no $x\leq b$ such that $f(x)=y$. (Note that whether a gap (a,b) in n is short does not depend on n.) A gap (a,b) of n is a *very short gap in n* if there is a $y\leq a$ for which there is an $x\leq\mu(n)$ with $f(x)=y$ but no $x \leq b$ for which $f(x) = y$. Note that we can computably determine the very short gaps in n but can only computably enumerate the short gaps in n.

For each n, we let $\mathrm{SG}(n)$ be the number of short gaps in n and we let $\mathrm{VSG}(n)$ be the number of very short gaps in n. As above, we can compute $\mathrm{VSG}(n)$ but in general can only approximate $\mathrm{SG}(n)$ in an increasing fashion as we discover the short gaps. We define our computable coloring by

$$c(n) = \begin{cases} \mathrm{VSG}(n) \bmod 2 & \text{if } i(n) = 1, \\ 2 + (\mathrm{VSG}(n) \bmod 2) & \text{if } i(n) = 2. \end{cases}$$

Let X be an infinite set such that $\mathrm{FS}^{\leq 3}(X)$ is monochromatic. We establish the following two properties.

(P5) For all $n, m \in X$, $i(n) = i(m)$.

(P6) For $k \geq 0$, there is at most one $n \in X$ such that $\lambda(n) = k$.

(P5) holds because $i(n) = 1$ implies $c(n) \in \{0, 1\}$ and $i(m) = 2$ implies $c(m) \in \{2, 3\}$. (P6) holds since if $n \neq m \in X$ with $\lambda(n) = \lambda(m)$ (and by (P5), $i(n) = i(m)$), then by (P2) and (P3), $i(n+m) \neq i(n)$ contradicting (P5).

By (P6), we can assume without loss of generality (by computably thinning out X) that if $n, m \in X$ with $n < m$, then $\mu(n) < \lambda(m)$. The argument now proceeds almost identically to the proof of Theorem 2.2 in Blass, Hirst and Simpson with one minor change.

First, we claim that for all $n \in \mathrm{FS}^{\leq 2}(X)$, $\mathrm{SG}(n)$ is even. For this claim, it is important that n is a sum of at most two elements of X. In particular, this claim need not hold for an arbitrary element of $\mathrm{FS}^{\leq 3}(X)$.

Fix $m \in X$ such that $n < m$, $\mu(n) < \lambda(m)$ and for all $y \leq \mu(n)$, if $y \in \mathrm{range}(f)$, then there is an $x \leq \lambda(m)$ with $f(x) = y$. Since n is a sum of at most two elements of X, $n + m \in \mathrm{FS}^{\leq 3}(X)$. Because $\mu(n) < \lambda(m)$, the gaps in $n + m$ consist of the gaps in n, the gaps in m, and the gap $(\mu(n), \lambda(m))$. We want to count the number of very short gaps in $n + m$. By the choice of m, the gap $(\mu(n), \lambda(m))$ is not very short in $n + m$. By (P4), $\mu(n + m) = \mu(m)$, so each gap in m is very short in $n + m$ if and only if it is very short in m. Finally, if (a, b) is a gap in n, then $b \leq \mu(n)$ and hence by the choice of m, (a, b) is very short in $n + m$ if and only if it is short in n. Therefore, we have

$$\mathrm{VSG}(n + m) = \mathrm{SG}(n) + \mathrm{VSG}(m).$$

Since $c(m) = c(n + m)$, the parity of $\mathrm{VSG}(m)$ is equal to the parity of $\mathrm{VSG}(n + m)$ and therefore $\mathrm{SG}(n)$ is even.

The last claim we need is that if $n, m \in X$ with $n < m$, then for all $y \leq \mu(n)$, $y \in \text{range}(f)$ if and only if there is an $x \leq \lambda(m)$ with $f(x) = y$. Note that this claim gives us a method to compute range(f) from X, completing the proof. To prove the claim, suppose for a contradiction that there is a $y \leq \mu(n)$ such that $y \in \text{range}(f)$ but there is no $x \leq \lambda(m)$ with $f(x) = y$. In this case, the gap $(\mu(n), \lambda(m))$ is short in $n+m$. Therefore, because the gaps of n (respectively m) are short in $n+m$ if and only if they are short in n (respectively m), we have

$$\text{SG}(n+m) = \text{SG}(n) + \text{SG}(m) + 1.$$

Since $n \neq m \in X$, we have $n+m \in \text{FS}^{\leq 2}(X)$ and hence SG(n), SG(m) and SG$(n+m)$ are all even, giving the desired contradiction. □

Formalizing the proof of this theorem in RCA_0, we obtain the following corollary.

Corollary 3.2. $\mathsf{RCA}_0 \vdash \mathsf{HT}_4^{\leq 3} \to \mathsf{ACA}_0$.

We now improve the previous theorem and corollary from 4 colors to 3 colors.

Theorem 3.3. *There is a computable coloring $c : \mathbb{N} \to 3$ such that if X is infinite with $FS^{\leq 3}(X)$ monochromatic, then $0' \leq_T X$.*

Proof. For any k and $i \in \{1, 2, 3, 4, 5, 6\}$, let $\mathcal{O}_{k,i} = \{n : n \equiv i \cdot 7^k \mod 7^{k+1}\}$. Let i_n denote the first nonzero heptary bit of n, which occurs in the k_nth place, so that $n \in \mathcal{O}_{k_n, i_n}$. Color each $n \in \mathbb{N}$ red, green or blue as follows with the slash indicating a choice between two colors depending on whether VSG(n) is even or odd.

$$c(n) = \begin{cases} R/G & \text{if VSG}(n) \text{ is even/odd and } i_n \equiv \pm 1 \mod 7, \\ G/B & \text{if VSG}(n) \text{ is even/odd and } i_n \equiv \pm 2 \mod 7, \\ B/R & \text{if VSG}(n) \text{ is even/odd and } i_n \equiv \pm 3 \mod 7. \end{cases}$$

Let $X \subseteq \mathbb{N}$ be an infinite set such that FS$^{\leq 3}(X)$ is monochromatic. We claim that $X \cap \mathcal{O}_{k,i}$ cannot contain more than 2 elements. To prove this claim, assume that x, y, z are distinct elements of $X \cap \mathcal{O}_{k,i}$ and hence $x + y \in \mathcal{O}_{k,(2i \mod 7)} \cap \text{FS}^{\leq 3}(X)$ and $x + y + z \in \mathcal{O}_{k,(3i \mod 7)} \cap \text{FS}^{\leq 3}(X)$. Consider the following table of multiplication facts.

i	$2i \mod 7$	$3i \mod 7$
± 1	± 2	± 3
± 2	± 3	± 1
± 3	± 1	± 2

The table shows that $\mathrm{FS}^{\leq 3}(X)$ must contain elements from each of the sets $\mathcal{O}_{k,\pm 1 \mod 7}, \mathcal{O}_{k,\pm 2 \mod 7}$, and $\mathcal{O}_{k,\pm 3 \mod 7}$ (where $\mathcal{O}_{k,\pm 1 \mod 7} = \mathcal{O}_{k,1} \cup \mathcal{O}_{k,6}$ and similarly for the other sets). However, by the definition of the coloring c, it is not possible for a monochromatic set to intersect all three of these sets. Therefore, if $x, y, z \in X \cap \mathcal{O}_{k,i}$ are distinct, then $\mathrm{FS}^{\leq 3}(X)$ is not monochromatic, proving the claim.

By the claim, if $\mathrm{FS}^{\leq 3}(X)$ is monochromatic, then X must include elements n for which k_n is arbitrarily large. Also, we can computably thin X so that all of its elements n share the same value for i_n and thus share the same coloring convention, guaranteeing a common parity for $\mathrm{VSG}(n)$. From here, we proceed as in the proof of the previous theorem. □

Corollary 3.4. $\mathsf{RCA}_0 \vdash \mathsf{HT}_3^{\leq 3} \to \mathsf{ACA}_0$.

4. Open Questions

Some of the open questions involve comparing bounded versions of Hindman's Theorem with special cases of Ramsey's Theorem. As usual, let RT^n_k denote Ramsey's Theorem for k-colorings of n-element sets. Thus, RT^n_k asserts that whenever the n-element subsets of $\mathbb{N}$ are k-colored, there is an infinite set $X \subseteq \mathbb{N}$ such that all n-element subsets of X have the same color.

We have provided some lower bounds on the strength and effective content of some versions of Hindman's Theorem for bounded sums. However, we do not know any upper bounds for the effective content and strength of $\mathsf{HT}^{\leq n}_k$ for $n > 1, k > 1$ beyond those known from [2] for Hindman's Theorem itself. In particular, we do not know whether any of these bounded versions of Hindman's Theorem are provable in ACA_0, or whether any of them imply HT. We also do not know whether $\mathsf{HT}^{\leq 2}_2$ implies ACA_0 in RCA_0, or whether Ramsey's Theorem for 2-coloring of pairs RT^2_2 implies $\mathsf{HT}^{\leq 2}_2$ in RCA_0.

One might also consider the restriction of Hindman's Theorem to sums of length exactly n. Let $\mathsf{HT}^{=n}_k$ denote the assertion that for each k-coloring $c : \mathbb{N} \to k$ there is an infinite set $X \subseteq \mathbb{N}$ such that $\{\sum F | F \subseteq X \text{ and } |F| = n\}$ is monochromatic. It is clear that RT^n_k implies $\mathsf{HT}^{=n}_k$ in RCA_0 for each $n, k \geq 1$, and indeed $\mathsf{HT}^{=n}_k$ is just the restriction of RT^n_k to colorings c of n-element sets F such that $c(F)$ depends only on $\sum F$. It follows from [8], Theorem 5.5, that each computable instance of $\mathsf{HT}^{=n}_k$ has a Π^0_n solution. It is unknown whether this result can be improved to Σ^0_n or better. It also remains open for each $n, k \geq 2$ whether $\mathsf{HT}^{=n}_k$ implies RT^n_k in RCA_0. We do not even know whether each computable instance of $\mathsf{HT}_2^{=2}$ has a computable solution.

Added Note (June 28, 2016): After this paper was submitted for publication, Denis Hirschfeldt pointed out to the authors that a result of Rumyantsev and Shen ([9], Corollary 2) can be used to give a quick proof that there is a computable instance of $\mathsf{HT}_2^{=2}$ with no Σ_2^0 solution. Indeed, he and Barbara Csima had used the same result from [9] to obtain a similar result with subtraction in place of addition. The details of the argument and further results will appear in a future paper.

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